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WORKS ON THEORY OF A FLAPPING WING

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## WORKS ON THEORY OF A FLAPPING WING

V.V. Golubev

### Theory of Boundary Layers<sup>1</sup>

One reason for the lack of any developed theory of the thrust of flapping wings is the following: if it is required that, as vortices are shed as it flaps and they are removed by the flow, the sharp trailing edge of the wing be the shedding point (basic Chaplygin-Zhukovskiy hypothesis), it must be assumed that the circulation of a vortex entrained by the flow changes with distance from the wing, but this contradicts the basic theorems of vortex theory. This first was met in the studies of N.Ye. Zhukovskiy on detachment of vortices, but it evidently went unnoticed. The authors of works on the theory of wings oscillating in an airflow attempted to avoid the purely theoretical difficulties which arose here, by assuming that an undulating vortex layer forms beyond the wing during its oscillation. This assumption is sufficiently consistent with experimental data, in the case of low amplitude oscillations during wing vibrations. Mainly the results obtained in this case were applied to questions of wing vibration. However, in extreme, large amplitude flapping of a wing, there apparently are no experimental data, which indicate the formation of a continuous vortex layer beyond the wing, which extends beyond the wing in the form of an undulating vortex surface. On the contrary, vortex formation of the Karman vortex street type, formed by separately moving vortices, can be observed beyond an oscillating wing. All the difficulties pointed out remain, for example, in the case of formation of a so called initial Prandtl vortex, in a sharp change of the angle of attack; in this case, available experimental data also do not give any basis for any conclusions as to the existence of a continuous distribution of vortices beyond the wing.

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The goal of this article is to show that the contradiction pointed out above can be removed, if the effect of the boundary layer of the wing is taken into account. The basis of further considerations is the systematic application of the theory of boundary layers to explanation of the processes which occur, during oscillations of a wing, in its boundary layer. Vortex formation processes can be explained by the viscosity of the air which, as is known, from the point of view of the theory of boundary layers, only appears within the boundary layer. From this point of view, the basic hypothesis that the sharp trailing

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<sup>1</sup> Published in the anniversary collection *Dvadtsat'let VVA RKKA im. Zhukovskogo* [20 years of the Military Air Academy of the Peoples' and Peasants' Red Army im. Zhukovskiy], 1942 (1st. ed.).

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\*Numbers in the margin indicate pagination in the foreign text.

edge is shedding point of the stream, the so called Chaplygin-Zhukovskiy hypothesis, is original only by lucky allowance for air viscosity. Frequently, the kinematic expression of this hypothesis, as a hypothesis of the physical impossibility of the existence of flows with infinitely high velocities, hardly makes any sense in application to an ideal fluid; in the case of the total absence of friction in the fluid, a limit can hardly be set for the value of the flow velocity.

From the point of view of the theory developed in this work, changes in structure of the boundary layer are the regulator, which permits, without violating the basic theorems of vortex theory in an ideal fluid, explanation of satisfaction of the Zhukovskiy hypothesis, in shedding from the wing of vortices with a constant circulation, which does not change during movement.

In all that follows, we consider the boundary layer of a wing as a unique regulator in the flow around the wing. Evidently, from this point of view, only an approximate, roughly schematic theory of boundary layers is sufficient.

As to the point of view of usual concepts of the Prandtl theory of boundary layers, we describe wing profile  $L$  and profile  $L_1$  en-

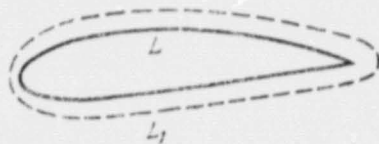
veloping it, which is the outer boundary of the boundary layer (Fig. 1). Beyond contour  $L_1$ , we have the potential flow, in which, because of the

small thickness of the boundary layer, we can consider the distribution of velocities and, consequently, pressures on contour  $L_1$ , which do not differ from the velocity and pressure distributions on contour  $L$ . Therefore, for the circulation, we can write the expression

$$\Gamma = \int_{L_1} \frac{dw}{dz} dz, \quad (1')$$

where  $w$  is the complex potential of the flow around the profile.

On the other hand, from the properties of the boundary layer, the flow velocity on  $L$  equals zero and, consequently,



$$\int_L \frac{dw}{dz} dz = 0. \quad (2')$$

Fig. 1.

It follows from these equations that

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$$\Gamma = \int_{L_1} \frac{dw}{dz} dz - \int_L \frac{dw}{dz} dz = \int_{L_1, L} \frac{dw}{dz} dz, \quad (3')$$

in which, as the  $L_1$ - $L$  boundary, we use the entire contour which circumscribes the region of the boundary layer (Fig. 2). Consequently,



Fig. 2.

from the kinematic point of view, we can replace the viscous boundary layer with a turbulent layer of an ideal fluid, in which its total vorticity equals the circulation around the wing. We base subsequent constructions on this boundary layer scheme. The vortices which form the boundary layer are similar in their properties to the vortices which are used in theories of the so called fluid wing: they are bound to the boundary layer region.

Consequently, any change in magnitude of the circulation around the wing causes the corresponding change in circulation of the boundary layer.

From the point of view of vortex theory, a system of vortices arranged around a rigid boundary should be accompanied by a system of vortices, which are the image of this system of vortices relative to a rigid wall; we can consider these vortices, either as purely fictitious vortices which replace the effect of the rigid boundary or, from the point of view of the theory of a fluid wing, as actual vortices in that mass of air which replaces the wing.

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Fig. 3.

Thus, we had the following two vortex zones in the wing: boundary layer zone I (Fig. 3) and a zone located inside the wing, which is the mirror image of zone I relative to the wing contour (zone II in Fig. 3).

With change in circulation around the wing, for example, with change in the angle of attack, the vorticity inside these layers changes. In this case, it also is possible that, with change in the flow past conditions, for example, with change in the angle of attack, part of the turbulent air mass which forms the boundary layer separates and forms a vortex leaving the wing; then, the symmetrical vortex in layer II remains inside the wing. In this case, the sign of the circulation departing with the vortex and the sign of the circulation which replaces the effect of the wall of the wing are opposite, and their absolute values are equal. Such a situation develops, for example, in the formation of the initial Prandtl vortex; we can consider it the result of separation of part of the boundary layer.<sup>2</sup>

There are similar circumstances in other cases, for example, in the shedding of vortices during flaps of a flapping wing or in a transient, for example, accelerated movement of a wing, which was discussed

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<sup>2</sup>We note that, in Prandtl photographs, it is very clearly seen how the departing initial vortex forms by twisting of the part of the boundary layer which is shed from the wing.



in the works of Wagner. In the latter case, from our point of view, the vortex track formed behind an equally accelerated moving wing is part of the boundary layer stripped from the wing surface. With such an opinion of the structure of the boundary layer and the origin of the circulation around the wing, it is natural that outside the boundary layer, i.e., in the range occupied by an ideal fluid, the Lagrange-Thomson theorem on the invariability of the circulation always will be satisfied but, within the boundary layer, these vortices can be produced since, from the physical point of view, we have a region in the boundary layer, which is occupied by a viscous fluid. Its replacement by a vortex layer is only of a purely kinematic nature. An example of such continuous formation of vortices within the boundary layer is the formation of a Karman vortex street behind the body past which the fluid flows. From our point of view, individual vortices of a Karman street are the result of breakdown of the boundary layer, in which individual parts of the layer shed form the vortices of a Karman street.

The presence of a vortex system within the boundary layer can show up during flow past, not only in the formation of the vortices shed from the wing, i.e., such vortices as Karman vortices or the initial Prandtl vortex.

The vortex distribution within the boundary layer can affect, for example, the position of the shedding point.

It is known from the theory of wings that, for total determination of the lift or its moment, the exact distribution of the circulation on individual elements of the wing surface need not be known. For example, in the theory of wings of finite span, to derive the basic formulas, it is sufficient to replace the system of connected vortices with one carrier vortex of variable circulation along the wingspan. In entirely the same manner, for further study, the boundary layer can be replaced by some approximate, quite rough scheme, but which is suitable for estimation of the total effect due to the boundary layer. From the point of view reported above, the boundary layer and its mirror image are, in the first approximation, the system of vortices indicated in Fig. 4, where L is the wing contour, I is the boundary layer region and II is the region occupied by the mirror image of the boundary layer. Because of the small thickness of the boundary layer, the distance between the vortices of systems I and II is very short and, therefore, the effect of two such vortex systems can, with some approximation, replace the effect of a system of dipoles, the axes of which are located along the wing walls. Thus, we replace the vortex system indicated in Fig. 4 by a system of dipoles (Fig. 5). Since the effect of such a system of dipoles, continuously distributed over the wing surface, can be replaced by the source and the flow at the ends of the boundary layer, on condition that the intensity of all continuously distributed dipoles is constant, in the first approximation, we can replace the effect of the boundary layer and its mirror image by a source and flow, located at two points of the wing surface. Such a substitution assumes a constant distribution of vortices in the boundary layer and arbitrary introduction of a "start" and "end" of the boundary layer. Therefore, such a simplifying assumption, of course, is very rough, but it presents no difficulties to making this scheme more flexible, on the one hand, by introducing a vortex system in the boundary

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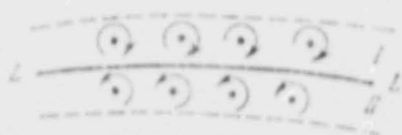


Fig. 4.

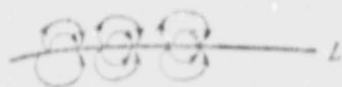


Fig. 5.

layer, which more closely coincides with the experimental data, in the distribution of the velocities produced by them in the boundary layer and, on the other hand, without replacing them, by means of the limiting transition by the dipoles, sources and flows.

A general conclusion from the preceding discussions is the following position: from the kinematic point of view, the effect of a boundary layer located on the wing surface can be replaced by a source and flow of some suitable intensity.

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We apply these considerations to study of the effect of a vortex near a cylinder on the flow past a round cylinder. On the assumption that we have flow past the cylinder with circulation  $\Gamma$  in the presence of a vortex of intensity  $J$ , for the characteristic function of flow past, we obtain the following expression:

$$w = Ve^{-i\theta} \left( z + \frac{R^2 e^{2i\theta}}{z} \right) + \frac{\Gamma}{2\pi i} \ln z + \frac{J}{2\pi i} \ln \frac{z - \rho e^{i\theta}}{z - \frac{R^2}{\rho} e^{i\theta}}, \quad (1)$$

where  $R$  is the cylinder radius,  $V$  is the flow velocity far from the cylinder,  $\theta$  is its angle to the effective axis and  $\rho e^{i\theta}$  is the affix of vortex  $J$ .

From this,

$$\frac{dw}{dz} = Ve^{-i\theta} \left( 1 - \frac{R^2 e^{2i\theta}}{z^2} \right) + \frac{\Gamma}{2\pi i z} + \frac{J}{2\pi i} \left( \frac{1}{z - \rho e^{i\theta}} - \frac{1}{z - \frac{R^2}{\rho} e^{i\theta}} \right). \quad (2)$$

We assume that the shedding point of the stream is point  $z=R$ ; we then obtain

$$Ve^{-i\theta} (1 - e^{2i\theta}) + \frac{\Gamma}{2\pi i R} + \frac{J}{2\pi i} \left[ \frac{1}{R - \rho e^{i\theta}} - \frac{1}{R - \frac{R^2}{\rho} e^{i\theta}} \right] = 0. \quad (3)$$

We find such a line that, by placing  $J$  at some point on it, we obtain equation (3) with constant  $\Gamma$ ,  $J$ ,  $V$  and  $\theta$ . For this line, from equation (3), we obtain the equation

$$\frac{1}{R - \rho e^{i\theta}} - \frac{1}{R - \frac{R^2}{\rho} e^{i\theta}} = \frac{2iV \sin \theta - \frac{\Gamma}{2\pi i R}}{\frac{J}{2\pi i}} = -\frac{4\pi VR + \Gamma}{JR}. \quad (4)$$

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If this reasoning is applied to the case when the vortex is shed from the cylinder, in which the intensity of the vortex and the circulation around the cylinder are constant and the shedding point of the jet does not shift, as the experimental data show, the vortices recede to infinity; by assuming  $\rho \rightarrow \infty$  in equation (4), we obtain

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$$\frac{1}{R} = \frac{4\pi V R}{H} \quad (5)$$

Then, from (4),

$$\frac{\left(\rho - \frac{R^2}{\rho}\right) e^{2i\theta}}{R^2(1 - e^{2i\theta}) - R\left(\rho + \frac{R^2}{\rho}\right) e^{2i\theta}} = \frac{1}{R} \quad (6)$$

From equations (5) and (6), we have

$$\frac{\Gamma - J}{\rho - \frac{R^2}{\rho}} = -4\pi V R, \quad (7)$$

$$\frac{2R \cos \mu - \left(\rho + \frac{R^2}{\rho}\right)}{2R \cos \mu - \left(\rho + \frac{R^2}{\rho}\right)} = -1.$$

From the latter equation (7), we obtain

$$\rho \cos \mu = R \text{ or } x = R.$$

Thus, under the problem conditions, the vortex should move perpendicular to the x axis, along tangent LL to the circumference (Fig. 6).

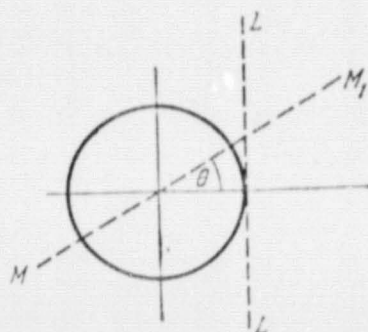


Fig. 6.

This result clearly contradicts both theoretical considerations and experiment. Both show that the vortices in the flow are displaced along the lines of the current, i.e., under the problem conditions, along a line parallel to lines  $MM_1$ . We obtain the conclusion

from this: a line cannot be selected, along which the vortex is displaced in free flow, in such a manner that the shedding point remains fixed, with constant circulation  $\Gamma$  and intensity of vortex  $J$ .

We attempt to satisfy the constant conditions by introducing the effect of the boundary layer. We take the layer in the simplified form which was pointed out above. The characteristic function has the

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$$w = Ve^{-i\theta} \left( z + \frac{R^2 e^{2i\theta}}{z} \right) + \frac{\Gamma}{2\pi i} \ln z + \frac{J}{2\pi i} \ln \frac{z - \frac{R^2}{\rho} e^{2i\theta}}{z - \frac{R^2}{\rho} e^{2i\theta}} + \frac{K}{2\pi} \ln \frac{z - Re^{i\theta}}{z - Re^{i\theta}} \quad (8)$$

where K is the strength of the fictitious source and stream replacing the boundary layer, and  $Re^{i\theta}$ ,  $Re^{i\theta}$  are their affixes.

From this,

$$\frac{dw}{dz} = Ve^{-i\theta} \left( 1 - \frac{R^2 e^{2i\theta}}{z^2} \right) + \frac{\Gamma}{2\pi i} \frac{1}{z} + \frac{J}{2\pi i} \left[ \frac{1}{z - \frac{R^2}{\rho} e^{2i\theta}} - \frac{1}{z - \frac{R^2}{\rho} e^{2i\theta}} \right] + \frac{K}{2\pi} \left[ \frac{1}{z - Re^{i\theta}} - \frac{1}{z - Re^{i\theta}} \right] \quad (9)$$

and for shedding point  $z=R$ , we obtain the equation

$$Ve^{-i\theta} (1 - e^{2i\theta}) + \frac{\Gamma}{2\pi i R} + \frac{J}{2\pi i} \left[ \frac{1}{R - \frac{R^2}{\rho} e^{2i\theta}} - \frac{1}{R - \frac{R^2}{\rho} e^{2i\theta}} \right] + \frac{K}{2\pi} \left[ \frac{1}{R - Re^{i\theta}} - \frac{1}{R - Re^{i\theta}} \right] = 0. \quad (10)$$

By noting that, from condition V,  $\Gamma$ ,  $J$ ,  $\theta$  are constant and by assuming, as above,  $\rho = \infty$ , and by considering that, at  $\rho = \infty$ ,  $K=0$ , we obtain

$$Ve^{-i\theta} (1 - e^{2i\theta}) + \frac{\Gamma}{2\pi i R} - \frac{J}{2\pi i R} = 0; \quad (11)$$

From this,

$$\Gamma - J = -4\pi VR.$$

From equations (10) and (11) we have

$$\frac{J}{2\pi i} \left[ \frac{\left( z - \frac{R^2}{\rho} \right) e^{2i\theta}}{R^2 (1 + e^{2i\theta}) - R \left( z + \frac{R^2}{\rho} \right) e^{2i\theta}} + \frac{1}{R} \right] + \frac{K}{2\pi R} \frac{e^{2i\theta} - e^{2i\theta}}{1 - (e^{2i\theta} + e^{2i\theta}) + e^{(2i\theta + 2i\theta)}} = 0.$$



From this,

$$J \left[ \frac{\beta - \frac{K^2}{\beta}}{2R \cos \beta - \left( \beta + \frac{K^2}{\beta} \right)} + 1 \right] +$$

$$+ K \frac{e^{\frac{\alpha-\beta}{2}} - e^{-\frac{\alpha-\beta}{2}}}{e^{\frac{\alpha-\beta}{2}} + e^{-\frac{\alpha-\beta}{2}} - \left( e^{\frac{\alpha-\beta}{2}} + e^{-\frac{\alpha-\beta}{2}} \right)} = 0$$

or

$$J \frac{2 \frac{K^2}{\beta}}{2R \cos \beta - \left( \beta + \frac{K^2}{\beta} \right)} + K \frac{2 \sin \frac{\alpha-\beta}{2}}{2 \cos \frac{\alpha-\beta}{2} - 2 \cos \frac{\alpha-\beta}{2}} = 0,$$

Finally, we have

$$K \frac{\sin \frac{\alpha-\beta}{2}}{\sin \frac{\alpha}{2} \sin \frac{\beta}{2}} = J \frac{R^2}{\beta^2 - 2R\beta \cos \beta + R^2}. \quad (12)$$

Equation (12) shows that, with any change in the values of  $\rho$  and  $\mu$ , i.e., with any movement of vortex  $J$ ,  $K$ ,  $\alpha$  and  $\beta$  can be selected, so that equation (10) is fulfilled, i.e., by regulation of the boundary layer, it always can be ensured that the shedding point of the stream on the cylinder remains fixed, with movement of a vortex of constant intensity  $J$  and with constant circulation  $\Gamma$  (or  $\Gamma-J$ ) around the cylinder.

The resulting conclusion shows that, in the processes which occur in transient movement, the role of the boundary layer is not limited to only the development of friction and the conditions which cause shedding of the stream from the wing surface. Apparently, allowance for the effect of the boundary layer is the only possible way of constructing a theory of flapping wings which does not contradict the basic conditions of hydromechanics.

## Thrust Development Mechanism of a Flapping Wing<sup>1</sup>

Despite the large number of studies on the theory of transitional movements of a wing, in particular, on the theory of a flapping wing, the hydrodynamic pattern of the development of thrust by a flapping wing is highly uncertain. The basic reason for this must be considered, in our opinion, the complexity of the hydrodynamic scheme which is the basis of all modern theory of wings, which are considered to be under conditions of transitional movement and continuously changing circulation velocity around the wing profile. /409

The basic difficulty which has to be met in constructing theories of an oscillating or flapping wing is the following. In the flaps of wings, just as in changes of rate of movement or in changes of the angle of attack, the wing circulation changes. Fulfillment of one of the basic conditions of vortex theory, the so called Thomson theorem, requires that, in change of wing circulation  $\Gamma$  by a certain amount  $\Delta\Gamma$  a vortex is shed from the wing with circulation  $-\Delta\Gamma$ , which, in being shed from the wing, is carried away further by the flow. On the other hand, the basic condition on which all modern theory of wings is constructed, the Chaplygin-Zhukovskiy hypothesis, is that the sharp trailing edge of the wing is the shedding line of the flow from the upper and lower surfaces of the wing. But, the position of the shedding point of the flow from the upper and lower surfaces of the wing depends on the magnitude of the circulation around the wing, as well as on the magnitude and positions of the external vortices. With change in position of the external vortices, the shedding point can remain on the trailing edge of the wing, only on condition of continuous change in the circulation which, in turn, according to the Thomson theorem, causes shedding of vortices from the wing. Based on this, Prandtl proposed the following hydrodynamic scheme: behind the wing which is in transitional movement, a vortex sheet forms, which is continuously shed from the trailing edge of the wing. Such a vortex sheet is a velocity discontinuity surface. Thus, a wing in transitional movement is accompanied by a velocity discontinuity surface which is shed from the trailing edge of the wing. /410

Such a velocity discontinuity surface or, which is the same thing, vortex surface, has a certain effect on the wing, as in any vortex system which affects, for example, the velocity field in which the wing is located. Taking account of this effect presents tremendous theoretical difficulties.

Various simplifying assumptions on the structure of the discontinuity surface must be introduced into the theory of transitional movement. For example, it usually is assumed that such a discontinuity surface is a plane. However, such a simplification, which is permissible in extremely low amplitude wing flaps, such as, for example, wing vibrations, proves to be completely unsuitable in the case of a flapping wing with large amplitude flaps.

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This hydrodynamic scheme of the phenomenon presents great difficulties, in accounting for the magnitude of the thrust of a flapping wing and in determination of the thrust development mechanism itself.

In current theories of flapping wings, the development of thrust usually is explained by the formation of suction at the leading edge of the wing. However, such an explanation appears doubtful, for the following reasons. As is known from the studies of N.Ye. Zhukovskiy, suction forces are generated at a sharp leading edge of a wing, but flapping wings, for example, of birds, just like the aerodynamic profiles of aircraft wings, have rounded outlines at the leading edge, i.e., they have a shape which is unsuitable for the development of thrust.

More than that, the construction of a theory of flapping wings, in particular, explanation of the mechanism of thrust development, is of undoubted technical interest. The fact is that, despite the most enormous progress of modern aviation, the question of flight economy of modern aircraft remains far from solved. It is highly possible that the energy used in flight by means of flapping wings proves to be considerably less than that used in modern aircraft, where the enormous flight speeds are responsible for the colossal power consumption and fuel consumption of the engine. The question of the comparative economy of modern flight vehicles based on the use of flapping wings is up against the lack of any developed theory of flapping wings. /411

An attempt is made in this article to report the basis of those considerations which, in our opinion, permit construction of a hydrodynamic theory of the operation of a flapping wing at any flap amplitude. We will consider the problem under plane parallel flow conditions.

As was pointed out above, the basic difficulty of the theory is accounting for the formation of the vortexes shed from the wing. According to the idea of Prandtl, used by all investigators of this question, a continuous vortex sheet is shed from a flapping wing. The basis of the proposed theory is the rejection of such a hydrodynamic scheme. We see a way towards agreement with those requirements to which the Chaplygin-Zhukovskiy postulate leads, on the one hand, and the theorem of the preservation of circulation (Thomson theorem), on the other hand, in accounting for the effect of the boundary layer of the wing, i.e., from the physical point of view, accounting for the effect of viscosity.

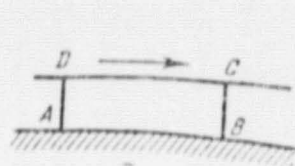


Fig. 1.

From the kinematic point of view, the boundary layer of a wing must be considered as a region filled with a turbulent fluid. Let AB be the surface of a streamlined body and CD be the outer surface of the layer (Fig. 1). Because of viscosity, the velocity on surface AB equals zero and, on CD, it equals the velocity of the flow past the streamlined body. Therefore, the circulation velocity in profile ABCD differs from zero, which is an indication of the vorticity of the fluid inside the boundary layer. The following kinematic model of a boundary layer can be imagined. We take some cylinders, with a diameter equal to the thickness of the boundary



layer, which roll along the surface of a solid, in such a manner that their rate of movement equals half the velocity of the flow along the outer boundary of the boundary layer (Fig. 2). After complicating this scheme somewhat, it is easy to imagine a kinematic model, which gives a velocity distribution within the boundary layer, which corresponds to the experimental data.

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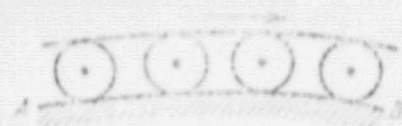
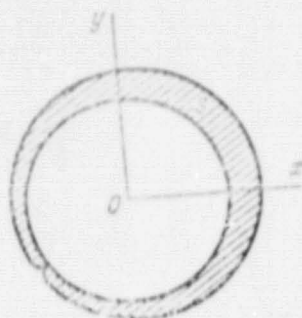


Fig. 2.

In order to determine what effect the boundary layer vortices have on the conditions of flow past the wing, we represent, as it usually is done in theory of wings, a region filled with a flowing fluid on the outer part of a circular cylinder of radius  $R$ , in such a manner that the sharp trailing edge of the wing is represented by point  $z=R$ . Here, the turbulent boundary layer of the wing is represented by vortex layer  $S$  around the cylinder (Fig. 3).

Let the vortex density in the layer be  $\sigma$ ; the characteristic function of flow past the cylinder can be written in the form



$$w = V_{\infty} e^{-i\theta} \left( z + \frac{k^2 z^2}{z} \right) + \frac{\Gamma}{2\pi i} \ln z + \frac{1}{2\pi i} \int_S \int z \ln \frac{z - re^{i\phi}}{z - \frac{k^2}{r} e^{i\phi}} dx dy, \quad (1)$$

Fig. 3.

where  $V_{\infty}$  is the flow velocity at infinity,  $\theta$  is the angle of attack,  $\Gamma$  is the circulation around the cylinder,  $re^{i\phi}$  is the affix of a vortex of a layer with vortex density  $\sigma$  and  $\frac{R^2}{r} e^{i\phi}$  is the affix of the representation of this vortex relative to the cylinder surface.

Since point  $z=R$  is the image of the sharp trailing edge, to fulfill the Chaplygin-Zhukovskiy postulate, it is necessary that

$$\left( \frac{dw}{dz} \right)_{z=R} = 0,$$

i.e.,

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$$V_{\infty} e^{-i\theta} (1 - e^{2i\theta}) + \frac{\Gamma}{2\pi i} \frac{1}{R} - \frac{1}{2\pi i} \frac{1}{R} \int_S \int \frac{r^2 - k^2}{r^2 - 2Rr \cos \frac{\theta}{2} + R^2} dx dy = 0, \quad (2)$$

from which

$$\Gamma = -4\pi R V_{\infty} \sin \theta + \int_S \int \frac{r^2 - k^2}{r^2 - 2Rr \cos \frac{\theta}{2} + R^2} dx dy. \quad (3)$$

As formula (1) shows, the boundary layer vortices do not produce circulation directly, since they are included in pairs of opposite sign, at points  $re^{i\theta}$  and  $\frac{R^2}{r}e^{i\theta}$ , but they have an indirect effect on the circulation, on the strength of fulfillment of the Chaplygin-Zhukovskiy postulate (since they affect the position of the shedding point of the flow from the wing).

In theories of wings, it is customary to use

$$\Gamma = -4\pi R V_\infty \sin \theta, \quad (4)$$

which is equivalent to the hypothesis

$$\oint_S z \frac{r^2 - R^2}{r^2 - 2Rr \cos \varphi + R^2} dx dy = 0. \quad (5)$$

We assume now that there is vortex J around the cylinder at  $r_1 e^{i\alpha}$ . The characteristic function of the flow takes the form

$$\begin{aligned} w = V_\infty e^{-i\theta} \left( z + \frac{R^2}{z} \right) + \frac{V}{2\pi i} \ln z + \frac{J}{2\pi i} \ln \frac{z - r_1 e^{i\alpha}}{z - \frac{R^2}{r_1} e^{-i\alpha}} + \\ + \frac{1}{2\pi i} \oint_S z \ln \frac{z - r e^{i\theta}}{z - \frac{R^2}{r} e^{-i\theta}} dx dy. \end{aligned} \quad (6)$$

In fulfillment of the Chaplygin-Zhukovskiy postulate, we obtain, similar to the preceding

$$\begin{aligned} \Gamma = -4\pi R V_\infty \sin \theta + J \frac{r_1^2 - R^2}{r_1^2 - 2Rr_1 \cos \alpha + R^2} + \\ + \oint_S z \frac{r^2 - R^2}{r^2 - 2Rr \cos \varphi + R^2} dx dy. \end{aligned} \quad (7)$$

From this, we see that, upon shedding vortex J, i.e., with change in  $r_1$  and  $\alpha$ , circulation  $\Gamma$  changes. This, as was pointed out in the

beginning, contradicts the condition of preservation of circulation (Thomson theorem). But, it is easy to show that conditions of change of the vortices within the layer (where, of course, the Thomson theorem cannot be valid, because of the development of viscosity) can be constructed, under which circulation  $\Gamma$  and J do not change in any position of vortex J. First and foremost, if vortex J recedes to infinity, from (7), we find

$$\Gamma = -4\pi R V_\infty \sin \theta + J \oint_S z \frac{r^2 - R^2}{r^2 - 2Rr \cos \varphi + R^2} dx dy, \quad (8)$$

where  $\sigma_\infty$  is the vortex density of the layer when vortex J is at infinity and, consequently, its image is in the center of the cylinder.

We find from equations (7) and (8) that, in fulfillment of the condition

$$\int_S \int (1 - \epsilon^2) \frac{r^2 - R^2}{r^2 - 2Rr \cos \alpha + R^2} dx dy + J \frac{2R(r_1 \cos \alpha - R)}{r_1^2 - 2Rr_1 \cos \alpha + R^2} = 0 \quad (9)$$

the circulation around the cylinder (Γ-J) remains constant at any position of vortex J, in fulfillment of the Chaplygin-Zhukovskiy postulate.

Condition (9) can be satisfied by appropriate selection of density  $\sigma$ , in which, by using condition (5), which is normal in the theory of wings, it can be considered that

$$\int_S \int \epsilon^2 \frac{r^2 - R^2}{r^2 - 2Rr \cos \alpha + R^2} dx dy = 0. \quad (10)$$

For example, with consideration limited to the simplest kinematic model of a boundary layer indicated in Fig. 2, instead of the surface distribution of vortices in the boundary layer, a linear distribution must be used (along the center line of the vortex cylinders rolling along the surface of the wing). In this case, if it is considered that the distance of the vortex line from the center of a round cylinder equals  $R + \epsilon$ , where  $\epsilon$  is small, in place of (9), we obtain

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$$\int_{\phi_0}^{\phi_1} \epsilon^2 \frac{2\epsilon + \epsilon^2}{2 + 2\epsilon + \epsilon^2 - 2(1 + \epsilon) \cos \alpha} d\alpha + J \frac{2R(r_1 \cos \alpha - R)}{r_1^2 - 2Rr_1 \cos \alpha + R^2} = 0,$$

or, by disregarding  $\epsilon^2$ ,

$$\epsilon^2 \frac{\sin \frac{\phi_1 - \phi_0}{2}}{\sin \frac{\phi_0}{2} \sin \frac{\phi_1}{2}} + J \frac{2R(r_1 \cos \alpha - R)}{r_1^2 - 2Rr_1 \cos \alpha + R^2} = 0,$$

where  $\phi_0$  and  $\phi_1$  are angles which correspond to the beginning and end of the vortex line which substitutes for the boundary layer.<sup>2</sup>

A general summary of all the preceding discussions is as follows:

<sup>2</sup>

I pointed out this simplest scheme in the article "Theory of Boundary Layers," anniversary collection Nauchnaya konferentsiya VVA KA [Scientific Conference of Red Army Military Air Academy], 1942 [see present publication, pp 1-8].

for fulfillment of the Chaplygin-Zhukovskiy postulate in departure from a wing of the vortices shed from it, it is not necessary to introduce a velocity discontinuity surface continuously shed from the trailing edge of the wing; the effect of change of position of the vortices at the shedding point of the flow from the wing is compensated by a change in structure of the boundary layer or, which is the same thing, by the effect of the viscosity of the air.

We now apply the preceding considerations to the following simplest scheme of a flapping wing.

We consider a flap sheet of width  $b$ , which is performing oscillations of finite amplitude  $h$ , under plane parallel flow conditions. Let the flow velocity at infinity be  $V_\infty$ , and we will consider that the velocity of the wing in lowering and raising the wing ( $+w$  and  $-w$ ) remains constant during lowering and raising, changing sharply at the extreme points of the oscillations.

Let, in the absence of flapping of the wing, the angle of attack equal  $\theta$ ; then, in the absence of flapping,  $\Gamma$  is determined by the formula

$$\Gamma = \pi b V_\infty \sin \theta. \quad (11)$$

In lowering the wing, an additional relative velocity is obtained, /416 directed upward and equal to  $w$ . In this case, the angle of attack increases by an amount  $\text{arctg} \frac{w}{V_\infty}$ , and the velocity of the windstream takes the form  $\sqrt{V_\infty^2 + w^2}$ .

During wind flapping, at the extreme points of the oscillation, vortices are shed from the wing, which can cause a change in circulation but, if the preceding considerations are taken into account, the effect of these departing vortices on the circulation can be disregarded.

Under these conditions, the circulation during lowering  $\Gamma_0$  is determined by the formula

$$\Gamma_0 = \pi b \sqrt{V_\infty^2 + w^2} \sin \left( \theta + \text{arctg} \frac{w}{V_\infty} \right). \quad (12)$$

If the axes are located as shown in Fig. 4, circulation  $\Gamma$  will be negative. Therefore, for  $\Gamma_0$ , we obtain the final expression

$$\Gamma_0 = -\pi b \sqrt{V_\infty^2 + w^2} \sin \left( \theta + \text{arctg} \frac{w}{V_\infty} \right); \quad (12')$$

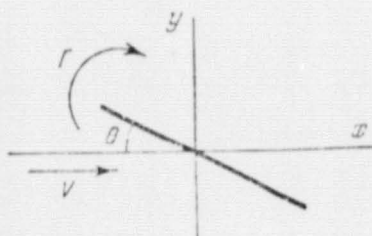


Fig. 4.

in exactly the same way, for the case of raising the wing, circulation  $\Gamma_r$

$$\Gamma_r = -\pi b \sqrt{V_\infty^2 + w^2} \sin \left( \theta - \text{arctg} \frac{w}{V_\infty} \right). \quad (13)$$



Consequently, in the transition from lowering to raising the wing, i.e., at the lower point of the oscillation, the circulation changes by an amount  $\Gamma_P - \Gamma_O$  which, according to the Thomson theorem, causes shedding from the wing of vortex  $\gamma_1 = -(\Gamma_P - \Gamma_O)$ . From equations (12) and (13), we have

$$\gamma_1 = -2\pi b \omega \cos \theta. \quad (14)$$

In just the same way, at the upper point of the oscillation, vortex  $\gamma = -(\Gamma_O - \Gamma_P) = -\gamma_1$  is shed from the wing, i.e.,

$$\gamma = 2\pi b \omega \cos \theta. \quad (15)$$

Thus, in wing flapping, at the upper and lower points of the oscillation, vortices of intensities  $\gamma$  and  $\gamma_1$  are shed from the wing.

Consequently, a vortex street forms behind the wing, which is similar to the Karman streets which form beyond barriers, but with vortex rotation in the opposite direction (Fig. 5).

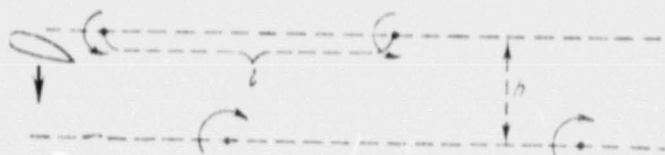


Fig. 5.

It can be assumed that the width of the street is  $h$ . With the period of oscillation of the wing designated  $T$ , the velocity imparted by the vortices of the street to all the vortices of the street by  $u_0$  and the distance between two successive vortices by  $l$ , we obtain

$$\left. \begin{aligned} l &= (V_\infty + u_0) T, \\ 2h &= \omega T. \end{aligned} \right\} \quad (16)$$

As follows from the known Karman formulas, we use

$$u_0 = \frac{\gamma}{2l} \operatorname{th} \frac{h\pi}{l},$$

or, by substituting the value of  $\gamma$  from (15), we obtain

$$u_0 = \Pi \frac{b}{l} \omega \cos \theta \operatorname{th} \frac{h\pi}{l}. \quad (17)$$

Since, subsequently, it is more convenient to have the ratio  $b/h$ , determined by the construction of the flapping wing, the following also can be written

$$u_0 = \frac{b}{h} \cos \theta \cdot \omega \left[ \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} \right]. \quad (18)$$



By excluding  $T$  from equation (16), we obtain

$$\frac{h}{h_0} = \frac{\frac{w}{V_\infty} - 2 \frac{h}{l}}{2 \frac{h}{l} \left[ \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} \right] \cos \theta \frac{w}{V_\infty}} \quad (19)$$

Since, obviously,  $\frac{b}{h} > 0$ , for the production of the street, the following condition must be satisfied

$$\frac{w}{V_\infty} > 2 \frac{h}{l}. \quad (20)$$

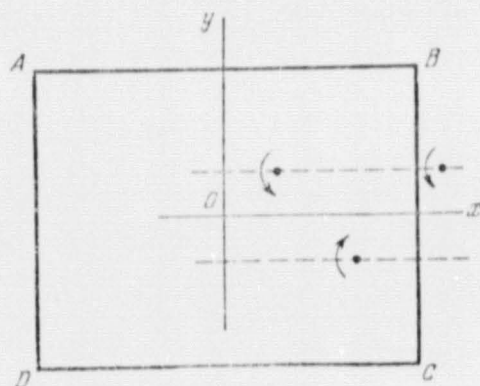
In particular, if the street satisfies the Karman stability condition,  $\frac{h}{l} = 0.281$  and  $\operatorname{th} \frac{h\pi}{l} = \frac{1}{\sqrt{2}}$ . Then, equality (19) takes the form

$$\frac{h}{h_0} = \frac{\frac{w}{V_\infty} - 0.562}{0.35 \cos \theta \frac{w}{V_\infty}} \quad (21)$$

and

$$\frac{w}{V_\infty} > 0.562. \quad (22)$$

All the preceding permits us to derive the basic equations of motion of a flapping wing. For this, following Karman, we write the equation of momentum and apply it to the mass of air included within a certain control surface, as which we use a rectangular parallelepiped, plotted on a square with very large sides. Let the center of the parallelepiped be at the coordinate origin, and the sides be parallel to the coordinate axes. In this case, we will assume that the wing oscillates around the center of the square, parallel to the  $y$  axis (Fig. 6).



We calculate the impulse of the force acting on such an air mass in oscillating period  $T$ . Let the pressure of the fluid on the wing give a force with components  $X, Y$ . Then, the wing acts on the fluid with a force with components  $-X$  and  $-Y$ .

Further, all the fluid outside the control surface acts on the fluid included in the volume under consideration. The corresponding pressure components are

$$X' = - \int_L p dy, \quad Y' = \int_L p dx. \quad (23)$$

Fig. 6.

The components of the force and pressure change periodically as the wings flap. Therefore, the components of the impulse of the force have the form

$$\left. \begin{aligned} \text{along x axis} & - \int_0^T X dt - \int_0^T dt \int_L p dy, \\ \text{along y axis} & - \int_0^T Y dt + \int_0^T dt \int_L p dx. \end{aligned} \right\} \quad (24)$$

We calculate the momentum of the fluid flowing through L in time T. By designating  $\theta$  as the angle formed by the velocity with the normal external to L, we obtain an expression for the amount of fluid flowing through element ds in time dt

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$$\rho ds V_1 dt \cos \theta = \rho dt (u dy - v dx).$$

From this, the components of the momentum have the form

$$\left. \begin{aligned} \text{along x axis} & \rho dt (u dy - v dx) u, \\ \text{along y axis} & \rho dt (u dy - v dx) v. \end{aligned} \right\}$$

Consequently, the increment of momentum in time T has the components along the axes

$$\left. \begin{aligned} \text{along x axis} & \rho \int_0^T dt \int_L (u dy - v dx) u, \\ \text{along y axis} & \rho \int_0^T dt \int_L (u dy - v dx) v. \end{aligned} \right\} \quad (25)$$

Further, two vortices pass through side BC in period T. The momentum carried away by them has the components along the axes:  $\rho \gamma h$  along the x axis and 0 along the y axis.

By applying the theorem of momentum, we obtain the following two equations

$$- \int_0^T X dt - \int_0^T dt \int_L p dy = \rho \int_0^T dt \int_L (u dy - v dx) u + \rho \gamma h, \quad (26)$$

$$- \int_0^T Y dt + \int_0^T dt \int_L p dx = \rho \int_0^T dt \int_L (u dy - v dx) v. \quad (27)$$

By multiplying both sides of equation (26) by 1 and adding the corresponding sides of (26) and (27), after some transformations, we obtain

$$\begin{aligned} & \int_0^T (Y + iX) dt = \\ & = -\rho \int_0^T dt \int_L \left[ (u dy - v dx)(u + v) + \frac{\rho}{2} (dx - i dy) \right] - \rho_0 h l. \end{aligned} \quad (28)$$

With the velocity potential designated by  $\phi(x, y, t)$ , we obtain, for the Lagrange integral, the expression

$$p = p_0(t) - \frac{\rho}{2} (u^2 + v^2) - \rho \frac{\partial \phi}{\partial t}, \quad (29)$$

and, since

$$\int_L p_0(t) (dx - i dy) = 0$$

and, also,

$$\int_0^T dt \int_L \frac{\partial \phi}{\partial t} (dx - i dy) = 0,$$

because of the periodicity of function  $\phi$ , equation (28) takes the form

$$\begin{aligned} \int_0^T (Y + iX) dt = & - \int_0^T dt \int_L \left\{ \rho (u dy - v dx)(u + v) + \right. \\ & \left. + \frac{\rho}{2} (u^2 + v^2) (dx - i dy) \right\} - \rho_0 h l. \end{aligned}$$

Since the velocity of the flow at infinity is  $V_\infty$ , by assuming

$$u = V_\infty + u'$$

and noting that

$$u'^2 - v^2 - 2iu'v = (u' - iv)^2 = \left( \frac{dw}{dz} \right)^2,$$

where  $w$  is the characteristic function of that flow which is superimposed on the basic flow with velocity  $V_\infty$ , because of the oscillations of the wing and, besides, that

$$\begin{aligned} \int_L u' dx + v dy &= \int_L (V_\infty + u') dx + v dy = \\ &= \int_L u dx + v dy = \Gamma, \end{aligned}$$

where  $\Gamma$  is the circulation around the wing and, because of the continuity of the flow

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$$\int_L u' dy - v dx = \int_L u dy - v dx = 0,$$

we reduce equation (28) to the form

$$\begin{aligned} \int_0^T (Y + iX) dt = \\ = -\rho V_\infty \int_0^T \Gamma dt - \frac{\rho}{2} \int_0^T dt \int_L \left( \frac{dw}{dz} \right)^2 dz - \rho \gamma h l, \end{aligned} \quad (30)$$

Also, by introducing the mean force with components  $X_m$ ,  $Y_m$  and the corresponding mean circulation  $\Gamma_m$ , we reduce equation (30) to the form

$$Y_m + iX_m = -\rho V_\infty \Gamma_m - \frac{\rho}{2} \left[ \int_L \left( \frac{dw}{dz} \right)^2 dz \right]_m - \frac{\rho \gamma h l}{\Gamma}. \quad (31)$$

To calculate the integral

$$\int_L \left( \frac{dw}{dz} \right)^2 dz$$

we note that the characteristic function of the flow consists of:

1. the characteristic function of the windstream with velocity  $V_\infty$ , i.e.,

$$w_1 = V_\infty z,$$

2. the characteristic function of the circulation flow which develops around the wing

$$w_2 = \frac{\Gamma}{2\pi i} \ln z,$$

3. the characteristic function of the flow generated by the vortices of the street behind the wing

$$w_3 = \gamma(z),$$

in which, on side BC, we will consider the street infinite at both ends;



4. to all the preceding, still another correction must be added, due to the presence of the wing in the flow (instead of a single vortex) and its movement during flapping, as well as the fact that the vortex street to the left of BC is finite. /423

If all these corrections are combined in residual term  $R(z)$ , the characteristic function of the entire flow takes the form

$$\omega' = V_\infty z + \frac{\Gamma}{2\pi i} \ln z + \varphi(z) + R(z), \quad (32)$$

from which

$$u - iv = V_\infty + \frac{\Gamma}{2\pi i} \frac{1}{z} + \frac{d\varphi}{dz} + \frac{dR}{dz}, \quad (33)$$

in which the order of magnitude of  $\frac{dR}{dz}$  on the sides of square ABCD, as well as the order of magnitude of  $\frac{d\varphi}{dz}$  on sides AB, CD and DA is

$$\frac{1}{|z|^2}.$$

By disregarding the flow with velocity  $V_\infty$  in the resulting formula, we finally obtain

$$\frac{dw}{dz} = u' - iv = \frac{\Gamma}{2\pi i} \frac{1}{z} + \frac{d\varphi}{dz} + \frac{dR}{dz}. \quad (34)$$

It is known from the theory of Karman streets that

$$\varphi = \frac{\Gamma}{2\pi i} \ln \frac{\sin(z - z_1) \frac{\pi}{l}}{\sin(z - z_2) \frac{\pi}{l}}, \quad (35)$$

where  $z_1$  and  $z_2$  are the affixes of one vortex of the upper row and one vortex of the lower row.

By substituting expression (34) in the integral  $\int_{BC} \left(\frac{dw}{dz}\right)^2 dz$ ,

we show that this integral differs from integral  $\int_{BC} \left(\frac{d\varphi}{dz}\right)^2 dz$  by a

term of the order  $\frac{1}{|z|}$ .

Consequently, by enlarging outline ABCD to infinity, we obtain

$$Y_m + iX_m = -\rho V_\infty l'_m - \frac{\rho}{2} \int_{BC} \left(\frac{d\varphi}{dz}\right)^2 dz - \rho \gamma h l \frac{1}{l}, \quad (36)$$

and, by using the method of calculation of the integral of the right side of the equality (36), which is conventional in the theory of Karman streets, we obtain the final formula

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$$Y_{cp} + iX_p = -\rho V_\infty \Gamma_{cp} - i \frac{\rho \gamma^2}{2\pi l} \left\{ \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1 \right\} - \frac{\rho \gamma h}{l} i,$$

where

$$T = \frac{l}{V_\infty + u_0}.$$

From this,

$$X_m = -\frac{\rho \gamma^2}{2\pi l} \left\{ \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1 \right\} - \frac{\rho \gamma h}{l} (V_\infty + u_0), \quad (37)$$

$$Y_m = -\rho V_\infty \Gamma_m. \quad (38)$$

Since

$$\Gamma_m = \frac{1}{2} (\Gamma_p + \Gamma_0) = -\pi \sqrt{V_\infty^2 + u_0^2} b \sin \theta \cos \operatorname{arctg} \frac{u_0}{V_\infty},$$

we finally obtain

$$\Gamma_m = -\pi V_\infty b \sin \theta,$$

i.e., the mean circulation equals the circulation in the absence of flapping.

By means of the equation

$$u_0 = \frac{\gamma}{2l} \operatorname{th} \frac{h\pi}{l}$$

equation (37) can be reduced to the form

$$X_m = \frac{\rho \gamma^2}{2\pi l} - \frac{\rho \gamma h}{l} (V_\infty + 2u_0). \quad (39)$$

This expression of mean thrust is completely analogous to the known formula for the drag given by Karman, and it is obtained from the Karman formula, if the signs of  $\gamma$  and  $u_0$  are changed.

In particular, if the condition of stability of the vortex street is substituted, i.e.,

$$\frac{h}{l} = 0.281, \quad \gamma = l \sqrt{8} u_0,$$

we obtain

$$X_m = -\pi V_\infty^2 \left[ 0,314 \left( \frac{u_\infty}{V} \right)^2 + 0,794 \left( \frac{u_\infty}{V_\infty} \right) \right]. \quad (40)$$

Thrust expression (40) also is completely analogous to the known /425  
formula for drag given by Karman.

It is interesting to note here that the problem of determination of the thrust of a flapping wing is considerably simpler than determination of the drag. As is known, in the case when we seek the drag, there are no theoretical ways to determine the intensity of the departing vortices  $\gamma$ . In the case of a flapping wing, we easily find the value of  $\gamma$  from formula (15).

By using formula (15), the following expression can be given for the drag

$$X_m = -\pi \rho b \omega V_\infty \cos \theta \frac{2 \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1}{\frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l}} \times \left\{ \frac{\omega}{V_\infty} - 2 \frac{h}{l} \frac{\frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1}{2 \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1} \right\}. \quad (41)$$

in which, in the general case, we find the value of  $h/l$ , according to the constructive and kinematic data from solution of equation (19):

$$\frac{b}{h} = \frac{\frac{\omega}{V_\infty} - 2 \frac{h}{l}}{2 \frac{h}{l} \left[ \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} \right] \cos \theta \frac{\omega}{V_\infty}}.$$

Finally, in the event the mode of flight is such that the stability condition is fulfilled, the following extremely simple expression is obtained for the thrust

$$X_m = -0,94 \pi b V_\infty \omega \cos \theta \left[ 1 + 1,2 \frac{\omega}{V_\infty} \right]. \quad (42)$$

It is extremely likely that the most suitable mode of flight is that when the condition of street stability is fulfilled.

In summary, we note that, as has been shown, accounting for air viscosity by introduction of the boundary layer of the wing simplifies the theory of flapping wings extremely. The possibility of dropping the introduction of continuous discontinuity surfaces, with their substitution by vortex streets, permits solution of the problem of the flapping wing with any flap amplitude. It is clear from the preceding that the theory can be constructed in the same manner, in the case of a wing of any profile and with any rule of change in velocity of the wing flapping  $w$ .

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## Theory of Flapping Wings and the General Problem of Thrust and Drag<sup>1</sup>

In one of his works, always somewhat unexpected and of enigmatic content, the recently deceased writer A.S. Grin (probably known to all from the ballet *Alyye parusa* [Red Saila] on its subject) describes the following scene. It takes place in a circus. A previously unknown actor, who has to demonstrate something unusual, emerges as the performer in one number on the program. The most ordinary man entered the arena and stood in the middle of the arena. Everyone looked and waited. . . suddenly, a little boy, sitting in the first row, bent down and shouted:

"He has risen from the ground!"

And everyone bent down and looked. It turned out that the man in the middle of the arena had separated from the ground and stood, not on the ground, but above the ground. . .

The audience, gathered 36 years ago, in the fall of 1908 at the race track at Khodynke, to watch the flight of Utochkin, one of the first Russian aviators, turned out to be in approximately the same situation. True, it was not necessary to bend down, to see that Utochkin rose in the air. He rose to the height of the second story, 6-8 meters, and he flew, in his clumsy Farman, which was similar to a large box kite, 150 meters along the stands, and he then landed on the ground. Essentially, this was not even flight. The pilot simply made a jump and, then, he moved above the ground for some time. /427

The enthusiasm of the audience was indescribable. The crowd ran towards the aircraft. They swung Utochkin and shouted "Hurrah." All the newspapers wrote about Utochkin, and no wonder: in the flights of Utochkin, the citizens of Moscow saw that man had separated from the earth and flown in a large, heavy machine.

Various records were then set: Latham rose to 300 meters above the earth; someone passed a swallow in flight; Bleriot flew across the English Channel. . . It was important to take the first step, to take off. Subsequently, it happened by itself, daring pilots, the inventiveness of talented engineers and designers, and the calculations of scientists went into working out the scientific problems of aviation.

Forty years ago, before the first flights of the Wright brothers, Santos Dumont and Farman, artificial flight was a task of the future, a dream, a fantasy of the scientists, at which they laughed, as they laughed at Zhukovskiy in his time, engaged in "trifles" in the mechanical office of the university, with the assembly of paper mechanical butterflies and various shapes of aerial kites.

Man now has learned to fly remarkably. He flies immeasurably

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<sup>1</sup> Published in the collection *Obshcheye sobraniye Akademii* [General Assembly of USSR Academy of Sciences], 1944.

higher and faster than the birds. Flight speeds are reaching such that, in the very near future, the distance from Moscow to Leningrad could be covered in one hour in an aircraft. Of course, no bird flies like this.

A modern aircraft lifts tremendous loads: tens and hundreds of passengers, tens of armed soldiers, tons of cargo, bombs, gasoline; it lifts artillery weapons and even tanks. The first mechanical problem, to explain from where the lift of an aircraft is obtained, what is the mechanism of development of lift, of course, was raised to science. This problem was solved completely in the brilliant works of scientists, the founders of modern engineering aeromechanics: N.Ye. Zhukovskiy, academician S.A. Chaplygin, Ludwig Prandtl, Lanchester and others. Their theories, formulas and calculations make it possible for present day designers and engineers to design and build tremendous steel birds, which carry colossal loads on their stretched out wings. The theory of lift now is a completely final, classical branch of mechanics. /428

But, on looking at a modern aircraft, a different question naturally arises. We have learned to fly remarkably, but how economically do we fly? Here and now, one comparison arises, which is not very favorable to our modern aircraft. On the one hand, we have remarkable examples of flight in nature, the flight of birds. By flapping the wings, they produce lift and thrust. They fly noiselessly, with high speed and to very great distances, by flapping the wings. Alongside are our aircraft with their wildly howling engines, with propellers making thousands of revolutions per minute, with immovably stretched out wings; aircraft burning colossal supplies of the most valuable fuel, gasoline, in their engines. While we have learned to fly very well, the economy of the modern method of flying is under great doubt. Of course, a basic problem arises from this: to study various methods of developing thrust, the pulling force of an aircraft. Permit your attention to be engaged by some theoretical considerations of the problem of the development of aircraft thrust.

Here, I have to deal with considerations which are extremely far from modern aviation technology and, moreover, of considerations which possibly never will be used by technology. But, in fact, such is the task of science: to study new, sometimes doubtful and unreliable pathways. It is comforting here that, in the case of success, these new pathways open up the widest possibilities for technology but, in the case of failure, at least, reliable indications are obtained as to the direction not to go in attempts at an engineering solution of a problem.

First and foremost, the following is striking in the problem of the conditions and mechanism of thrust development: the engineering solution of the problem of thrust development by the use of a propeller, an airscrew, differs radically from the solution of the problem of thrust development taken by nature, where the thrust which is used in the flight of birds is developed in a completely different way, by flapping the wings.

It is completely clear that, for technology, the direct imitation of nature is optional and atypical. It is sufficient to remember the /429

engineering application of rolling, in the form of diverse wheels, cylinders, rollers, etc., beginning with the primitive araba and ending with the most sophisticated steam engines, the bearings of modern machines, automobiles, tracked mechanisms, etc. It also must be noted that, in nature, at least at first glance, rolling is hardly found. On the contrary, for the movement of animals, more or less complicated rod mechanisms are used, such as the legs of men and animals. I stated that rolling is "hardly" found in nature. Further, we shall see that there is a quite widespread mechanism in nature, where we have a kind of rolling, evidently in a much more perfect form than is achieved in technology.

In any case, the preceding considerations show that study of the question of the development of thrust by flapping wings is a problem which merits detailed and attentive study. Yet, if we turn to modern scientific theories, we find the unexpected fact that, while the mechanism of thrust development by the rotation of a propeller was studied in extreme detail in a number of works of various scientists, first and foremost, in the classical studies of N.Ye. Zhukovskiy, the mechanism of thrust development by flapping wings was not studied at all. As an example, I present two attempts to produce a solution of this problem.

In some studies on the theory of wings, thrust is explained by suction generated at the leading edge of the wing. This suction actually can occur, regardless of flapping, if the leading edge is sharp. Yet, alas, the leading edge of the wings of all birds is just not sharp, but rounded, as it is assumed to be on considerations of the general theory of wings but, moreover, such an explanation is connected in no way with wing flapping.

In a physics course, this explanation is given: "in flight, by means of beating the wings, birds actively produce lift. Here, the wing is somewhat twisted, so that its trailing portion is bent upwards. Because of this, the wing surface, oblique to the direction of motion, produces a forward impulse."<sup>2</sup> /430

Yet, the leading part also proves to be convex ahead and, with such reasoning, a backward pulling force should develop from it.

It can be said of these explanations that a drowning man grasps at straws: when there is no explanation even somewhat similar to reality, anything goes. Discussion of this question ends pessimistically, for example, in Aerodinamka [Aerodynamics] of Durand:

"Exact determination of thrust is difficult. We will not investigate this problem."<sup>3</sup>

2

E. Grimzel', Kurs fiziki [Course in Physics], vol. 1, issue 1, State Theoretical and Technical Publishing House, 1933.

3

Durand, Aerodinamka [Aerodynamics], vol. 2, 1938, p. 347.



However, it must be noted that, in this latter book, the authors (Karman and Burgess) in our opinion, approach close to a correct understanding of the matter.

The goal of studies conducted by many of my colleagues in the Institute of Mechanics, Moscow State University, was to fill a gap in the theory of wings in this area. In the present report, I have tried to give a general idea of these studies, and I am attempting to familiarize you with the conclusions towards which these studies led.

When the development of thrust is concerned, first and foremost, the question arises, from where can this force develop. Mechanics teaches us that action equals reaction, that any force causes a force in the opposite direction. The force which ejects a projectile from a weapon causes the forces which produce the recoil, the kick of the weapon. This force is absorbed by the action of the recoil cylinder and the resistance of the earth. In a spring repelled from the earth, in addition, we have repulsion of the earth; only because of the colossal mass of the earth, it is completely insensitive to the thrust which we impart to the earth with the spring. An aircraft flying above the ground transmits pressure to the ground through the air and repels it in the opposite direction. The force which pushes a rocket forward is produced by the force of the exhaust gases ejected from the rocket by the burning mixture, etc. In a propeller, this backward deflection of the air is produced by the propeller blades which, by rotating, drive the air backwards, as a result of which, because of the law of action and reaction, the thrusting force of the propeller is produced. Essentially, all theories of propellers are reduced to that which explains how, behind a running propeller, the jet of air ejected by it is produced. Yet, while the deflection of air by a running propeller is more or less clear mechanically, the backward deflection of air by a flapping wing is completely uncertain, since the wing flaps in a direction perpendicular to the direction of motion and, consequently, to the direction of deflection of the air. /431

Another problem, in a certain sense, the reverse of the problem of the development of thrust, can answer this question. This is the problem of the development of the drag of bodies past which a stream flows. In fact, it is completely clear algebraically that the drag is negative thrust and, conversely, thrust is negative drag, precisely as profit is negative loss and loss is negative profit.

Thus, we imagine that some obstacle is fixed in a stream of air or fluid, which the stream flows around and which experiences more or less significant pressure from this stream. The abutments of a bridge in a river can be an example. The water flow presses against them. We ask ourselves what is occurring here and from where the pressure of the flow comes. In Fig. 1 and 2, we have photographs of what occurs in a stream when it flows around a barrier. We see that the smoothly flowing stream forms, at first glance, a quite disordered flow beyond the body. We now have to interpret what is occurring before us, and to understand the mechanical meaning of this complicated flow. /432

If we attempted, with complete accuracy, to describe theoretically the process occurring before us, we would not obtain any scientific

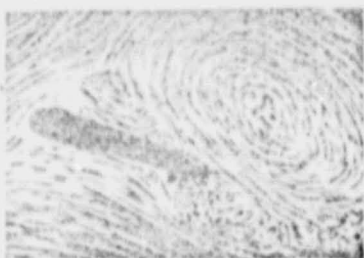


Fig. 1.

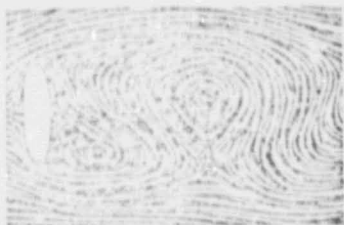


Fig. 2.

results at all. Yet, however, that is just the case, that science is not a photograph, which accurately transmits the transient and, generally speaking, random state of what is being described in all details. No! Science is a great artist and, like an artist, it simplifies and formalizes reality. As Levitan, in his landscapes, does not undertake the task of accurately picturing, in all details, all the leaves and twigs of the trees, as Repin, in his portraits, does not attempt to delineate all the wrinkles of the faces pictured by rejecting random and uncharacteristic details, to accurately transmit what is characteristic and general, which expresses the essence of the person depicted, in quite the same way, any scientific theory must discard random and insignificant details so as, by formalizing and simplifying the phenomenon, to reveal thereby its basic traits, those mechanical principles which govern it. We will not be surprised by this formalization. A cubist portrait by Picasso also transmits, to some extent, the nature and essence of the face depicted. The only criterion of value of any

scientific theory is, alone, experimental verification of the conclusions obtained from it.

Thus, in what manner does science formalize the picture of flow past obtained from tests?

The first try at such formalization was that of Helmholtz and Kirchhoff and, subsequently, it was developed with great success, in the works of a number of other scientists, particularly successfully in the works of Lord Rayleigh and N.Ye. Zhukovskiy. This is the so called theory of flow. We easily grasp the essence of this theory from the following observations, which are completely clear from everyday life. When a strong wind blows, we hide from it behind the corner of a building. When there is a strong current in a river, we can get away from it in a boat behind an island. The wind blows behind the corner and there is a current behind the island but just much weaker. Thus, we formalize the phenomenon! Behind the obstacle, instead of some complicated gyrating, but comparatively weak currents, assume we have a completely quiet zone, a region of aerodynamic or hydrodynamic shadow. Thus, instead of the actual flow pattern (Fig. 1 or 2), we will consider a schematic, stylized picture (Fig. 3 or 4), in which zone S, which is behind the barrier around which flow occurs, is filled with a completely stationary medium and, at boundary L, which separates the flow zone from the region occupied by the stationary liquid, the flowing liquid slides along the stationary liquid, quite the same way as a hand slides over a well polished table.

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As Helmholtz, Kirchhoff, Rayleigh, Zhukovskiy and others have shown, such a scheme can be subjected to exact mathematical calculation. In particular, Rayleigh gave a formula, by which the pressure



Fig. 3.

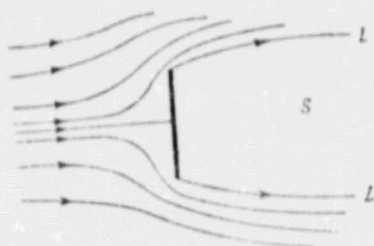


Fig. 4.

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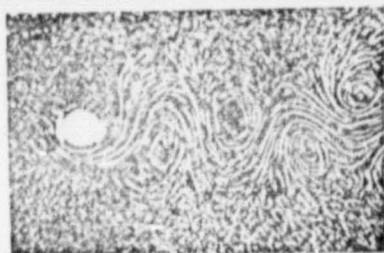


Fig. 5.

formations have long been the subject of systematic, detailed study. The most significant results were obtained by Benard and Karman. Therefore, such streets usually are called "Benard-Karman streets."

We now have to interpret mechanically what direct observation provides. Mechanics teaches us that such vortices only can form and disappear in liquids because of the viscosity of the liquid and that, in low viscosity liquids, i.e., not such as molasses, preserves, oil, etc., the effect of viscosity appears only right at the surface of the body around which flow occurs, within a certain, generally speaking, extremely thin, so called "boundary" layer. Outside it, the viscosity is so negligible that it can be considered, with sufficient accuracy, that the

of the flow at the barrier can be calculated. Yet, the theoretical value of the pressure obtained in this manner does not agree well with tests. It is approximately half that actually observed. From this, we obtain the conclusion:

The formalization of the flow past phenomenon on which the theory of flow is based is an extremely far from real picture of the phenomenon, and it cannot be considered to satisfactorily transmit the essence of the phenomenon.

If all our preceding reasoning is attentively looked over, it is easy to find the point, at which, as can be thought, we went too far from reality. This is those gyrating, turbulent movements of the fluid which we see behind the barrier and which we did not take into account at all in the theory of flow. In that, at first glance, chaotic flow which we observe behind the body around which flow occurs, these rotating, turbulent movements emerge completely distinctly. Moreover, if we go a little away from the body around which flow occurs, we note completely clearly that these vortices form a very orderly configuration beyond the body past which flow occurs. As the engineers say, they form a double vortex street, in which the vortices of both sides of this street are displaced from each other by half the distance between vortices, so that the vortices of the street are located, as they say, in checkerboard order (Fig. 5). Such vortex streets can be observed beyond the abutments of bridges over rivers, if only the flow is sufficiently rapid. For example, in Leningrad, such vortex streets can be observed very well at the Tuchkov Bridge.

Such a vortex street sometimes extends quite far because of the viscosity of the liquid, the vortices of the street gradually blur and, finally, they disappear. Such stable vortex



medium has no viscosity at all there. Among such media with extremely low viscosity are air and water. Thus, it follows from the foregoing that the reason for the formation of vortices beyond the body must be sought in the effect of viscosity in layers of liquid or gas immediately adjacent to the surface around which flow occurs, i.e., within the boundary layer.

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But, what occurs in the boundary layer?

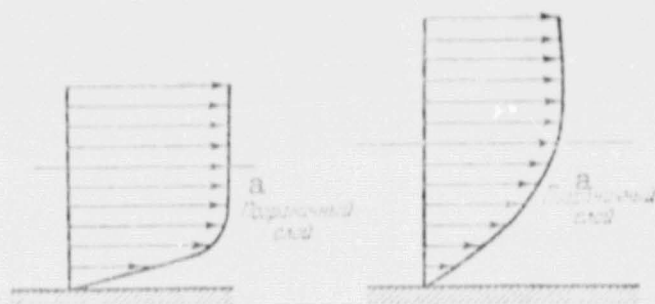


Fig. 6.

Key: a. boundary layer

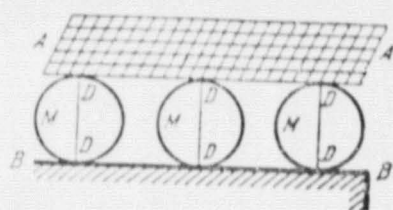


Fig. 7.

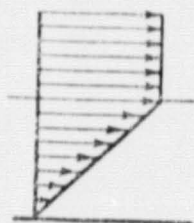


Fig. 7a.

Experience shows that a flowing liquid adheres to the wall and, consequently, is stationary relative to it and, with distance from the wall, the velocity gradually increases and, at a certain distance, it equals the velocity which the flow would have, if there were no viscosity at all. Graphs are given in Fig. 6, which show the velocity distribution within the boundary layer at various flow rates or, as it is called, which give "velocity profiles."

On the other hand, imagine that plate AA rolls on cylinders M over fixed plane BB (Fig. 7). If it is noted that the points of contact of cylinders M with the fixed plane are instantaneous centers of rotation and, consequently, have zero velocity, the velocity profile at the points of moving plate AA and the points on the diameter of the cylinder DD have the form shown in Fig. 7a. It is evident that, if the wheels are arranged in two rows, as shown in Fig. 8, in which plate AA moves on rollers M over plate B which, in turn, moves on cylinders  $M_1$  over fixed plane CC,

the corresponding velocity profile has the form shown in Fig. 8a. It is completely obvious that, by increasing the number of intermediate plates, we can approximate the profile of such a mechanism, composed of cylinders, as close as desired to the velocity profiles of a liquid within the boundary layer (Fig. 6). We obtain a most important conclusion from this: all particles of a fluid within a boundary layer can be considered in a state of rotational motion, i.e., the entire fluid is strongly turbulent.

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Consequently, the scheme given in Fig. 7 can be considered the simplest mechanical representation of what occurs within the boundary

layer. Also, the boundary layer is a sort of layer composed of cylinders, over which the remaining mass of liquid outside the boundary layer moves. Everywhere in nature, where there is movement of a liquid or gas around a fixed wall, a boundary layer is produced in some form and, consequently, we have a mechanism which is a kind of system of cylinders. Thus, by using any kind of wheel, roll, cylinder, etc. in engineering, we only extremely imperfectly imitate an incomparably more delicate mechanism, which is extremely widespread everywhere in nature, where there is flow with the formation of a boundary layer. /437

By using such a mechanical scheme, we now can very simply answer the question as to the source of the vortices which form Benard-Karman streets beyond an object around which flow occurs. We assume that stationary plane BB, along which rollers M roll (Fig. 7), ends. Then, the rollers which come off the edge of the plane leave it rotating. In quite the same manner, the vortices which form the boundary layer, upon being shed from the edges of the body around which flow occurs, move further beyond the body in the fluid. The case of liquid or gas flow is somewhat more complicated, only insofar as the vortex formations shed are entire more or less complicated systems of such cylinders and, moreover, the turbulent motion induced by them involves the entire mass of fluid, both in front of and behind the obstacle. This explains the extremely complicated, at first glance, system of vortex motions which we observe beyond a body around which flow occurs (Fig. 1 or 2).

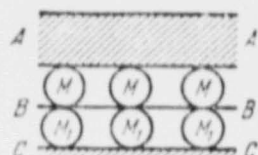


Fig. 8.

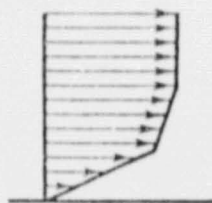


Fig. 8a.

Thus, we now can say that the vortices which form Benard-Karman streets are the breakdown of the turbulent boundary layer.

We still have to explain why, at some distance behind the body around which flow occurs, the vortices into which the continuous boundary layer breaks down form an orderly system of a double vortex, checkerboard street. This question was completely explained more than 30 years ago, in the works of Karman.<sup>4</sup> It turns out that, of all the vortex formations which can develop behind a body around which flow occurs, the only system which has an adequate degree of stability is a system which forms a double vortex checkerboard street. In this case, the arrangement of the vortices has to follow an exact rule, namely: if the distance between two successive vortices of a street is designated  $l$  and the width of the street is  $h$ , for stability of the street, fulfillment of the following condition is necessary

$$ch^2 \frac{h\pi}{l} = 2,$$

from which

$$\frac{h}{l} = 0.281.$$

<sup>4</sup> Th.v. Karman, "The fluid state and air resistance," Physik. Zeitschrift, issue 13, 1912.



Since, of course, only stable formations can exist in nature, systems of vortices observed behind bodies around which flow occurs satisfy this condition (Fig. 9).

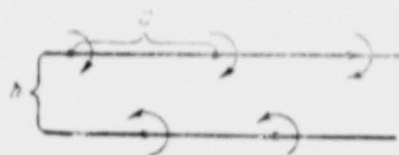


Fig. 9

From the physical point of view, it appears to me that the very fact of the unusual quantization of a turbulent medium upon breakdown of a boundary layer is extremely interesting, and that the method of approach to solution of the problem stated is highly similar to the solution of a number of problems of atomic theory and quantum mechanics, where questions of the stability of certain discrete states of matter play such an exceptional part.

We now present some results and, first and foremost, we try to represent schematically the process of vortex street formation. The simplest scheme, as follows from the preceding considerations, is the following: the boundary layer formed on the leading side of an obstacle around which flow occurs, from the mechanical point of view, is a series of air cylinders or rollers, by means of which the medium flowing past the barrier rolls; these rollers, which schematically represent the turbulent matter of the boundary layer, roll off the ends of the barrier at points A and B (Fig. 10), and they are carried further by the fluid, forming a double checkerboard vortex street in it, which satisfies the stability conditions.

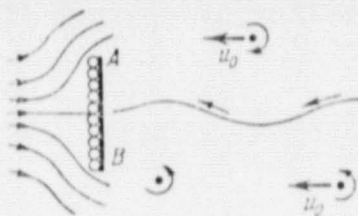


Fig. 10.

close to the barrier around which flow occurs, is determined by the formula

$$u_0 = \frac{\Gamma}{2l} \tanh \frac{h\pi}{l},$$

where  $\Gamma$  is the intensity of individual vortices of the street.

In a similar manner, it turns out that, by the vortices of a Benard-Karman vortex street, the entire mass of fluid flowing between the two vortex chains acquires velocity against the direction of the velocity of the initial flow, as is shown in Fig. 10.

We now have all the data for explaining the mechanism of formation of the thrust of a flapping wing. Further, we see that beyond a flapping

wing, such double checkerboard vortex streets form, in which the direction of the vortices is the reverse of the direction of rotation in Benard-Karman streets. We will call such a street an inverse Benard-Karman street.

The reason for all of this is as follows. The basis of modern theory of wings, the basic outlines of which were developed in the works of N.Ye. Zhukovskiy and Academician S.A. Chaplygin, is the mechanical idea that an aircraft wing can be considered to be situated in a vortex flow: this is the associated vortex of the wing, as N.Ye. Zhukovskiy called it. The intensity of this vortex depends, on the one hand, on the geometric properties of the wing cross section and, on the other hand, on the flight conditions. This is expressed mathematically by the /440 formula

$$\Gamma = \pi b V \left( \frac{\alpha}{2} + \theta \right),$$

where  $\Gamma$  is the vortex intensity,  $b$  is the chord (width) of the wing,  $V$  is the flight speed,  $\alpha$  is a quantity which characterizes the camber of the wing and  $\theta$  is the angle of attack. An explanation of all these quantities is given in Fig. 11. According to the famous formula of Zhukovskiy, this quantity  $\Gamma$  now determines the magnitude of the lift of the wing  $P$

$$P = \rho V \Gamma s,$$

where  $\rho$  is the air density and  $s$  is the wingspan. These formulas, obtained purely theoretically, are in good agreement with test data, at the small angles of attack which are used exclusively in aviation. An example of a test and theoretical graph of the change in intensity of the associated vortex of the wing as a function of the angle of attack is given in Fig. 12.

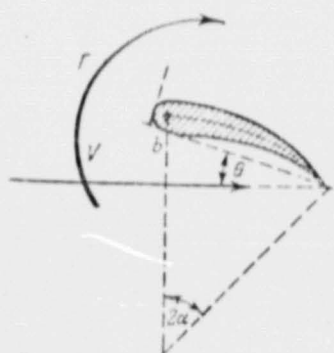


Fig. 11.

We now apply all these formulas to the case of a flapping wing. In order not to complicate the matter, we take the simplest case of a wing, a flat, rectangular, long plate, which is performing sharp oscillations upward and downward at velocity  $w$ . In this case, there is no camber and, therefore,  $\alpha=0$ , and the graph of change of  $\Gamma$  has the form indicated in Fig. 13. We consider two positions of the flapping wing: one, when the wing is lowered and the other, when the wing is raised upward (Fig. 14). When the wing descends, the velocity of the flow around the wing is made up of two velocities: velocity  $V$ , equal to the flight speed, but in the opposite direction, and velocity  $w$  perpendicular to it, directed upwards, since the wing is descending. Thus, the resulting velocity  $w_1$  forms angle of attack  $\theta_1$  with the wing, greater than the angle of

attack which there would be, if it were not for the velocity of descent of the wing. In a completely similar manner, for the case of raising the wing, we obtain resulting velocity  $w_2$ , composed of velocity  $V$  and

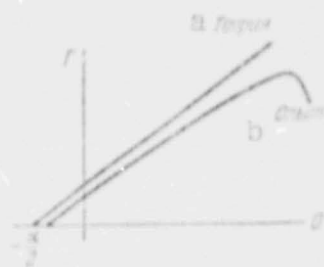


Fig. 12.

Key: a. theory  
b. test

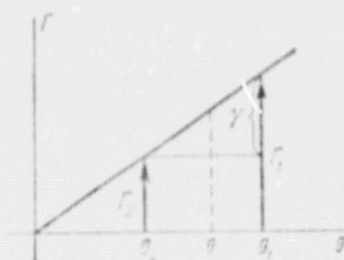


Fig. 13.



Fig. 14.

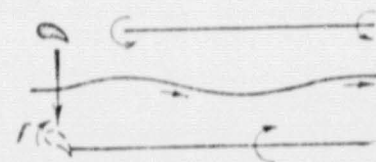


Fig. 15.

velocity  $w$ , directed downward, since the wing is being raised upward; in this case, the resulting velocity forms angle of attack  $\theta_2$ , less than  $\theta$ .

Turning now to the graph which gives the values of  $\Gamma$ , we see that intensity  $\Gamma_1$  of the associated

vortex of the wing is greater during descent than that during ascent of the wing  $\Gamma_2$ , by a certain

amount  $\gamma$  (Fig. 13). Thus, in the transition from raising to lowering the wing at the upper point of oscillation of the wing, the intensity of the associated vortex increases by amount  $\gamma$  and, in changing from lowering to raising the wing at the lower point of the oscillation, the intensity of the associated vortex decreases by amount  $\gamma$ . Yet, according to the basic theorems of the theory of vortices, such changes in circulation are possible, only in the event a vortex, with an intensity equal to the change in intensity of the associated vortex of the wing and opposite in sign, is shed from the wing in the flow. Thus, at the upper points of the oscillation, a vortex is shed from the wing, with intensity  $\gamma$  and directed opposite to the direction of rotation of the associated vortex and, at the lower points, a vortex is shed, also of intensity  $\gamma$ , but directed in the same

direction as the associated vortex (Fig. 15). Since these shed vortices are carried away by the flow, a double vortex street is formed beyond the wing, with a checkerboard arrangement of the vortices but, as is evident from the diagram, the direction of rotation of these vortices is opposite

to the direction of rotation in the Benard-Karman streets. Thus, we have the following basic result: beyond a flapping wing, an inverse Benard-Karman vortex street forms.

The result we have obtained now permits explanation of the development of thrust in a flapping wing and its magnitude to be found. In fact, the direction of rotation of the vortices of the resulting inverted street indicates that both the vortices themselves, and the entire mass of fluid

included between the two vortex chains receives additional velocity from them, directed back from the wing and, therefore, according to the classical laws of mechanics, a recoil force directed forward should act on the wing. This is thrust. In a quite similar manner, in the case of formation of a conventional Benard-Karman street, the velocity imparted to the fluid beyond a body around which flow occurs causes a recoil force directed with the flow, i.e., the pressure of the flow. In summarizing all that has been stated, we can say:



With the flaps of a flapping wing, we cause the development of an inverted Benard-Karman vortex street behind it which, in turn, produces a force directed opposite to that which develops from a conventional street, i.e., thrust.

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Fig. 16.

Such is the physical basis of the proposed theory of the generation of thrust of a flapping wing. We now deal with other attempts to solve this same problem. The basis of the proposed theory is the consideration that, at the end points of the oscillations, the vortices stripped from the wing are shed from it. Without going into detail here, we note that we also consider these vortices to be the result of breakdown of the boundary layer, quite similarly to the considerations presented above. Yet, at first glance, all these conclusions contradict

one postulate, which is quite precisely substantiated experimentally. In 1910, S.A. Chaplygin introduced one assumption, on which all modern theory of wings is based. This assumption, the basic hypothesis of the theory of wings, is that in smooth flow past a wing, the sharp edge of the wing is the shedding point of the stream (Fig. 16). Yet, it is easy to show that, if a vortex is shed from the wing, its movement causes, generally speaking, a displacement of the shedding point of the stream and, consequently, the Chaplygin hypothesis is not fulfilled. In order for the Chaplygin postulate to be fulfilled under such conditions, it had to be assumed that, in movement of external vortices, the circulation of the wing changes, which contradicts the basic theorems of vortex theory. Thus, in order to adopt the scheme which is reported above, it would appear to be either necessary to reject the experimentally well tested Chaplygin postulate, or to give up the basic theorems of vortex theory.

Authors working in this area have avoided these difficulties in this way: they assumed that, in the movement of external vortices, the Chaplygin postulate is fulfilled, and change in circulation occurs, because of the continuous shedding of vortices from the trailing edge of the wing, the result of which is the formation of a continuously trailing vortex sheet behind the flapping wing. However, it apparently can be considered experimentally proved that such a continuous sheet is not observed. Tests in the flow channel of the hydraulics laboratory of Moscow State University have shown quite convincingly the formation of discrete vortices behind a flapping wing, which form an inverted double vortex street. Thus, for substantiation of all the preceding theory, it had to be shown that there is such a mechanism on the wing, which permits satisfaction of the Chaplygin postulate, without subsequently disrupting the basic theorems of vortex theory.

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It can be shown that the boundary layer of the wing is such a mechanism. The vortices of which the boundary layer consists, as it is easy to see, do not affect the circulation of the wing, but they do affect the position of the shedding point of the stream. Thus, if the boundary layer is suitably changed in the shedding of external vortices from the street, reconciliation with the Chaplygin postulate, on the one hand, and the basic assumptions of vortex theory, on the other hand,



is easy. In regard to this, the mathematical considerations were given by me in a short note two years ago.<sup>5</sup>

We note here that, up till now, the boundary layer of a wing has been considered the source of development of the drag of the wing and that it has permitted explanation of the physical phenomenon of separation of the stream. From the point of view of the theoretical considerations developed here, the role of the boundary layer in explanation of the mechanism of flow of a fluid past a body turns out to be considerably more complicated. On top of everything noted above, the boundary layer is an unusual regulator of the process of shedding the stream from the wing. It is highly likely that, itself, the process of quantization of the vortices shed from a wing, which form vortex streets, is completely determined by the equilibrium conditions of the boundary layer, which we do not know at present.

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All the theoretical considerations reported above now permit formulas to be obtained, which give the value of the thrust. First and foremost, it is easy to show that this method, which Karman used for solution of the problem of drag, is utilized for calculation of the thrust of a flapping wing. Therefore, the final result is obtained in almost completely the same way. Karman gave this formula for the drag

$$W = \rho V^2 \left[ 0.794 \frac{u}{V} - 0.314 \left( \frac{u}{V} \right)^2 \right].$$

In the present case, the difference is that  $u$ , the velocity produced by the effect of the vortex street, changes sign, since the direction of rotation of the vortices and, consequently, the direction of the flow produced by these vortices, is the reverse. The sign and force  $W$  change in just the same way, since drag is superseded by thrust in the present case. From this, we obtain the following expression for the thrust

$$T = - \rho V^2 \left[ 0.794 \left( \frac{u_0}{V} \right) + 0.314 \left( \frac{u_0}{V} \right)^2 \right].$$

Subsequently, however, it turns out that determination of the thrust is considerably simpler than determination of the drag. In fact, velocity  $u_0$  is determined through the intensity of the vortices of the Karman street  $\gamma$  by the formula

$$u_0 = \frac{\gamma}{2l} \operatorname{th} \frac{\pi z}{l}.$$

<sup>5</sup>

V.V. Golubev, "Theory of boundary layers," anniversary collection Dvadtsat' let VVA RKKA im. Zhukovskogo [20 Years of the Zhukovskiy Military Air Academy of the Workers and Peasants Red Army], 1942 [see this publication, pp. 1-8].

In the case of determination of the drag, we do not now have any reliable theoretical ways to determine the value of  $\gamma$ . This is an extremely delicate problem, undoubtedly connected with the question of the stability of the turbulent boundary layer. It follows from this that the Karman formula does not make possible a full theoretical determination of the drag. The theoretical considerations developed above permit very simple determination of the value of  $\gamma$  in the case of a flapping wing. It is determined by the formula

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$$\gamma = 2\pi b\omega \cos \theta.$$

From this, it is easy to obtain a final expression of the thrust

$$T = -1.76 b\omega V \cos \theta \left[ 1 + 0.25 \frac{h}{b} \cos \theta \frac{V}{V'} \right].$$

All the preceding conclusions were obtained, on the assumption that the vortex street developed behind a flapping wing satisfies the stability conditions. In the general case of any street, we obtain the more general formula

$$T = -\pi b\omega V \frac{\frac{2h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1}{\frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l}} \left\{ \frac{V}{V'} - 2 \frac{h}{l} \frac{\frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1}{2 \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1} \right\} \cos \theta.$$

As the structure of the formula itself shows, the thrust can revert to zero or even change sign. It is of interest to note that the absence of thrust in a flapping wing occurs, only under conditions which are far from fulfillment of the stability conditions.

The possibility of the thrust reverting to zero in the presence of a vortex street perhaps explains the fact, which is difficult to explain mechanically, that hovering in the air without forward speed is possible with flaps of the wings, as can be observed in the flight of a dragonfly and, for an extremely short time, of birds which, by working the wings, can hover in one place. The necessity of fulfillment of a condition which, in any case, is close to fulfillment of the stability condition, for the development of thrust evidently explains the fact of hunting, well known to all, in which game frightened by firing, in attempting to escape from the firing, begins convulsively to flap the wings with extreme force, but it does not increase flight speed by this. From the point of view considered here, this is explained by the fact that, in this case, it departs from the flight conditions which produce thrust.

We now present some results. As it turned out, the task of determination of the thrust of a flapping wing led us to two, in a certain sense, opposite problems: the determination of thrust and the determination of drag. We now can supplement somewhat those considerations, on which the flow theory of Rayleigh drag is based. We saw that, in this theory, it is assumed that, at the interface, a moving mass of fluid slides along a fixed wall, as if along some

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completely smooth and hard wall. Yet, such a concept of the surface of the interface, as applied to a liquid or gas, is devoid of any physical meaning. In a liquid or gas, the velocity discontinuity surface only can be conceived as a vortex layer. Segard-Karman vortex streets are such a vortex layer, which breaks down because of the instability of the individual vortices. Yet, in particular, the conclusion is then obtained that additional energy consumption is necessary in formation of the vortices which replace the velocity discontinuity surface. This energy only can be obtained by increasing drag. Of course, the flow theory cannot take this latter portion of the drag into consideration.

We now turn to the general theory of drag. In all the foregoing, we did not consider the effect of the ends of the body around which flow occurs or the ends of the wing. Expressed in the language of mechanics, we considered the problem of thrust and drag in plane-parallel flow. We now proceed to the three dimensional problem.

We imagine that we have a certain body, around which a current flows. A hydrodynamic shadow zone also forms behind this body and, consequently, all the preceding reasoning can be transferred here. N.Ye. Zhukovskiy first attempted to construct a vortex theory of drag for this case. In all likelihood, he generated the idea of this theory, in connection with his work on the vortex theory of propellers. As is known, the basis of this theory is that a system of vortices forms behind a rotating propeller, which can be presented in simplified form as a vortex cylinder (Fig. 17). In this case, the vortices in it are arranged along the generatrix and along circumferences  $L, L, \dots$  and, moreover, axial vortex  $\Gamma$  forms. We will not consider the axial vortex or the vortices along the generatrix of the cylinder, since they all do not produce thrust, but produce only rotation of the medium behind the propeller.

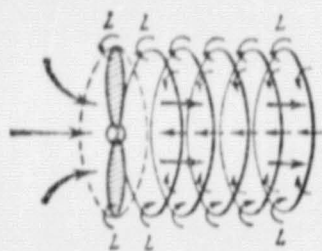


Fig. 17.

We consider the vortices on circumferences  $L, L, \dots$ . It is evident that they produce an additional velocity inside the cylinder, directed away from the propeller, i.e., they produce repulsion of the air backwards and away from the propeller and, therefore, because of the reaction, the propeller receives forward thrust.

If, on the other hand, a body, for example a circular disk around which a current flows is considered, we see that a system of vortices forms behind it, which are the breakdown of the boundary layer formed on the plate past which flow occurs.

A vortex surface also forms behind the body, which is similar to the surface behind the propeller, which is formed by vortices  $L, L, \dots$ , but the direction of rotation in these vortices is inverse to the direction of rotation of the vortices behind the propeller and, therefore, the additional flow velocity they produce now is directed against the velocity (Fig. 18). The result of this is the generation of forces along the direction of flow of the current, i.e., drag. Such is the vortex scheme of the phenomenon.

Unfortunately, at present, from this vortex scheme, not only could



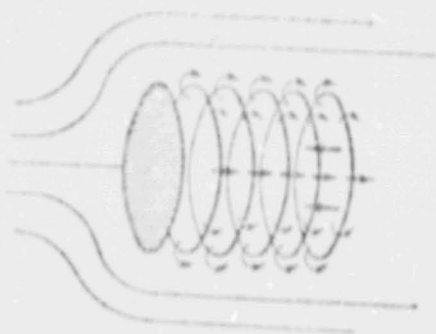


Fig. 18.

the value of the drag not be obtained theoretically, but we even do not have any clear picture of the velocity distribution behind the body past which flow occurs. Here, science has failed to produce a scheme which, in any way, satisfactorily explains the processes which occur in nature. More than that, the engineering importance of these studies is hardly clear to anyone. In fact, the tremendous power produced by the engines of modern aircraft overcome the basic drag, which is generated by the effect of diverse vortex systems shed from the aircraft wings and fuselage. We were engaged in study of all these questions during the war years at the Institute of Mechanics at Moscow State University. /449

In the foregoing, we repeatedly saw that these studies in various areas are connected with those ideas, which we find in the works of our famous founder of modern engineering aeromechanics, the "father of Russian aviation," N.Ye. Zhukovskiy. Not only in the scientific ideas do we attempt to continue the scientific traditions of N.Ye. Zhukovskiy. We are attempting to cultivate them and continue in the method of investigation. In all his brilliant scientific activity, N.Ye. Zhukovskiy showed how a clear physical scheme and precisely conducted experiment play a leading, fundamental part in mechanics. His scientific creativity showed that mechanics is, first and foremost, a natural science, the task of which is to study one of the simplest phenomena of nature, to study the motion of matter. This essence of mechanics, as a natural science, the science of nature, completely determines the tasks and method of mechanics.

Let these results on the mechanism of the generation of thrust, which we observe everywhere in nature, in the flight of birds and insects, which I had the honor and satisfaction of reporting to you in general outlines, be a modest gift of profound thanks for the activity of our notable naturalist, president of the Academy of Sciences of the USSR, V.L. Kolomarov, the important date in the renowned activity of whom we note in the present session. /450

ORIGINAL PAGE IS  
OF POOR QUALITY



## Thrust of a Flapping Wing<sup>1</sup>

In modern hydromechanics and aeromechanics, studies of questions of transient motion undoubtedly occupy a central place. The most diverse and extremely urgent problems of modern aviation, like the theory of acrobatics, the effect of individual wind gusts on a flying aircraft, the effect of an ascending air flow, accounting for diverse wing vibrations, beginning with the theory of flapping wings and ending with various wing vibration phenomena and the work of propellers are individual problems of the theory of transient motion of a fluid. The practical importance of studies in this field have led to the appearance of a very large number of works. In particular, in the USSR, we have had a series of excellent work on transient motion in the past decade. In a meeting of the USSR Academy of Sciences a few months ago, we heard an interesting paper by associate member of the USSR Academy of Sciences, A.I. Nekrasov, in which there was given a survey and analysis of numerous studies on the theory of wings in a nonstationary flow.

This paper, which deals with the same field, has a very much more modest aim, to report here the considerations on which the methods of solution of one particular problem of the theory of periodic motions of a wing are based.

In October 1944, I had the honor of reporting to a meeting of the general assembly of the USSR Academy of Sciences some most general physical considerations, on which the proposed theory are based. The purpose of this report is to substantiate these considerations and the particular conclusions to which they lead.

In modern hydromechanics, the theory of wings is a completely structured part of it, which studies smooth flow past bodies with a multiple value potential, in which the characteristic cyclic period of the polyvalence of the potential, the magnitude of the velocity circulation around the wing, has the basic role in the entire theory. Determination of the circulation values is based on the experimental assumption that the sharp trailing edge of a wing is the stream shedding line in smooth flow past the wing. This assumption, the so called Chaplygin-Zhukovskiy postulate, is the basis of the entire modern theory of wings. However, the physical essence of this postulate is not now completely clear. It can be thought that this postulate is an indirect allowance for the effect of viscosity in modern theory of wings, constructed on the basis of the theory of an ideal, i.e., completely devoid of viscosity, medium. /451

As calculations whisper, the magnitude of the circulation, through which the forces acting on the wing are expressed, depends partly on wing shape, partly on the velocity of the flow past the wing and on the orientation of the wing relative to the flow (angle of attack). An extremely important consequence for the entire theory of wings in nonstationary flow is obtained from this. With change in flow velocity

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Report to meeting of Engineering Sciences Section, USSR Academy of Sciences, 11-12 January 1946. Published in Izv. AN SSSR, Otd. tekhn. nauk 5, (1946) (1st ed.).

or in a change in angle of attack, generally speaking, the magnitude of the circulation around the wing changes, if the Chaplygin-Zhukovskiy postulate is fulfilled.

On the other hand, because of one of the basic theorems of vortex theory, the Thomson theorem, in an ideal fluid, the circulation over a given physical contour does not change during the flow (the principle of conservation of circulation). These two mutually contradictory assumptions in the theory of wings are reconciled, on condition that, with change in circulation around the wing of some magnitude, a vortex is shed from the wing into the flow past it, with a circulation equal to and opposite in sign to the change in circulation around the wing. Thus, the total circulation over a physical contour which envelops the wing and the vortex shed from it remains constant, as the theory of Thomson requires. We note here that the very formation of a vortex shed from the wing is physically impossible in an ideal fluid, i.e., without allowance for viscosity. Consequently, fulfillment of the Thomson theorem is possible, only by allowance for the viscosity of the medium. At present, the most convenient means of theoretical allowance for viscosity is the so called theory of boundary layers. From the point of view of this theory, departing vortices are the breakdown of a turbulent boundary layer. Such departing vortices can be observed extremely clearly in a test. For example, there are excellent motion picture films, which show the formation of departing vortices at the beginning of movement of the wing (the so called initial vortex).

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A consequence of all the reasoning presented above is that, in an ideal fluid, in which viscosity develops nowhere, present day circulation theory of wings does not occur. In it, in smooth flow past a body, we obtain nothing but the D'Alembert paradox. From the point of view of modern physical concepts, liquid helium is such an ideal fluid. From the point of view of the opinions developed here, there are no Zhukovskiy forces in liquid helium, the flow is noncirculatory and the D'Alembert paradox is exactly satisfied. It is completely natural that it would be extremely interesting to verify the conclusions by test.

The preceding considerations lead to the conclusion that, with continuous change in flight speed or angle of attack, vortices are shed continuously from the wing. These vortices are then carried away by the flow and form a continuous vortex sheet behind the wing [1]. Since the vortices, continuously distributed over the surface, produce an increase in velocity of the flow, on the one hand, and a decrease, on the other, this surface is a velocity discontinuity surface. Such velocity discontinuity surfaces are mentioned in all studies on motion of a wing in a nonstationary flow with changing circulation.

The presence of turbulent velocity discontinuity surfaces creates exceptional difficulties for the entire theory of a wing in nonstationary flow. The vortices which form it affect the wing and each other. The effect of these vortices on each other generally results in breakdown of the sheet, since it can be shown that such vortex formations are unstable. The breakdown of a vortex sheet results in the formation of some system of discrete vortices. The effect of the vortices on the vortices on the wing is reduced to the vortices changing the velocity around the wing, which complicates calculation of the forces acting on

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the wing extremely. Therefore, in theoretical investigations on motion of a wing in nonstationary flow, one or another assumption which simplifies the problem has to be introduced. Usually, these simplifications are in two areas.

First, the difficulties pointed out above drop out if, in nonstationary motion, the circulation remains constant, due to a suitable combination of conditions of motion of the wing. Such cases were noted first by S.A. Chaplygin in 1926. Subsequently, a number of other scientists were occupied with them. Despite the very great theoretical interest in such research, it must be remembered that they are of a quite particular nature. Of course, a change in circulation is the most characteristic fact in nonstationary motion and, by dropping consideration of it, we essentially "throw out the baby with the bath water" in this problem.

Second, simplifying assumptions can be made, relative to the form of the departing vortex sheet. Thus, for example, in wing vibration theory, because of the small amplitude of the vibration, it is assumed that the vortex sheet is a plane. This assumption, not very convincing at small amplitude but with not very flat waves, is completely impermissible in the case of large amplitude vibrations, which we have in the problem of the flapping wing. As a result of all this, in the case of a nonstationary flow around a wing with unchanged circulation, in theoretical investigations, we have almost exclusively results which concern the case of infinitely small oscillations, which limits the problem extremely.

I would like to emphasize here that the difficulties mentioned above are not of a mathematical, but of a purely physical nature. The matter consists of uncertainty of the physical scheme itself, which could formalize the processes occurring in the flapping of the wing, and the deficiency of this physical scheme cannot be replaced by any differential or integral equations or other mathematical means, however complex they might be.

The theory which I am attempting to set forth here is based on a certain physical scheme of the flapping wing phenomenon, which is completely different from that indicated above. In this case, we restrict ourselves to the case of a flapping wing somewhat artificially, but we transmit the basic physical scheme of the phenomenon more distinctly. We assume that the velocity of rise of the wing  $w$  and the velocity of its descent  $-w$  remain constant during raising and lowering. Subsequently, it is easy to determine that the same scheme, without any change, can be the basis of a theory of flapping wings, in the general case, when  $w$  changes by any rule. /454

The physical hypothesis which is the basis of the present theory is as follows: in flaps of a wing, a continuous vortex sheet does not separate from it but, at the upper and lower points of the oscillations, vortices are shed from the wing, which form a double vortex street of the Benard-Karman street type behind the wing.

First and foremost, we must show how the Chaplygin-Zhukovskiy postulate and the Thomson theorem can be satisfied in such a physical scheme. The fact is that the vortices of the street shed from the wing



change velocity at points on the wing surface and, consequently, the circulation changes continuously in fulfillment of the Chaplygin-Zhukovskiy postulate. In turn, this would have to cause continuous shedding of vortices, i.e., we come right back to the vortex sheet scheme.

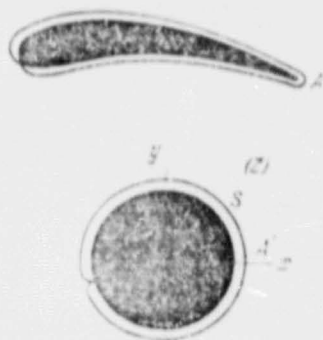


Fig. 1.

A way out of the difficulty produced can be found, by taking account of the effect of viscosity. The extremely fortunate thought of Prandtl, to reduce the effect of viscosity to the action of a boundary layer, from the kinetic point of view, has the result that the boundary layer can be considered a turbulent zone of an ideal fluid. Thus, the wing is not only exposed to the flow, but to the continuously distributed vortices which fill the boundary layer. We now use a conventional method in the theory of wings. We represent the flow region on the external part of a round cylinder by an auxiliary plane of the complex variable ( $z$ ) (Fig. 1). In this case, let the trailing edge of the wing correspond to point  $z=R$ . In this representation, the turbulent boundary layer around the wing changes to turbulent layer  $S$  around the surface of a cylinder.

By using the conventional method of reasoning, we obtain the following equation which, in the present case, satisfies the characteristic function of flow past a cylinder

$$w(z) = Ve^{-i\theta} \left( z + \frac{R^2 e^{2i\theta}}{z} \right) + \frac{\Gamma}{2\pi i} \ln z + \frac{1}{2\pi i} \int_S \int z \ln \frac{z - re^{i\varphi}}{z - \frac{R^2}{r} e^{i\varphi}} dx dy, \quad (1)$$

where  $V$  is the velocity at infinity,  $\theta$  is the angle it forms with the  $x$  axis,  $\Gamma$  is the circulation around the cylinder,  $re^{i\varphi}$  is the affix of the point of the turbulent zone with vortex density  $\sigma$ .

As is known, the applicability of the Chaplygin-Zhukovskiy postulate leads to requirement that the velocity at point  $z=R$  on the cylinder equal zero, i.e., in order to fulfill the equation

$$Ve^{-i\theta} (1 - e^{2i\theta}) + \frac{\Gamma}{2\pi i} \frac{1}{R} - \frac{1}{2\pi i R} \int_S \int z \frac{r^2 - R^2}{r^2 - 2rR \cos \varphi + R^2} dx dy = 0,$$

from which, for the circulation around the wing  $\Gamma$ , we obtain the expression

$$\Gamma = -4\pi R V \sin \theta + \int_S \int z \frac{r^2 - R^2}{r^2 - 2rR \cos \varphi + R^2} dx dy. \quad (2)$$



This formula differs from that normally used (without allowance for the vortices of the boundary layer)

$$\Gamma = 4\pi R V \sin \theta$$

by the term

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$$\iint \frac{r^2 - k^2}{r^2 - 2rk \cos \varphi + k^2} dx dy. \quad (3)$$

Thus, the usual derivation of the circulation requires that expression (3) equal zero, which does not require  $\sigma=0$ , i.e., the absence of a turbulent boundary layer.

As Eq. (1) shows, the vortices of the boundary layer do not themselves produce the circulation, since they are included by pairs with intensity  $+\sigma$  and  $-\sigma$ , but they affect the circulation indirectly, because of fulfillment of the Chaplygin-Zhukovskiy postulate, since they affect the position of the shedding point.

Eq. (2) shows that, with change of  $V$  and  $\theta$ , circulation  $\Gamma$  can remain constant, if the change of  $V$  and  $\theta$  are compensated by the corresponding change of  $\sigma$ , i.e., the vorticity of the boundary layer.

An equation analogous to (2) can be written, for the case when we have one or more departing vortices around the wing. Analogous reasoning leads us to the conclusion that the effect of the departing vortex on the circulation can be compensated by corresponding change of the vorticity of the boundary layer, so that, on the one hand, the Chaplygin-Zhukovskiy postulate will be fulfilled (by virtue of the fulfillment of Eq. (2)) and, on the other hand, the Thomson theorem will be fulfilled (by virtue of the constancy of  $\Gamma$ ). It is evident that such compensation of the change in flow by a change in the boundary layer is possible, only until these changes cause breakdown of the boundary layer.

Thus, the hypothesis on which the theory under consideration is based consists of the following. In the transition of the wing from the upper point of the oscillation to the lower and back, the effect of the departing vortices on the position of the shedding point is compensated, with constant  $\Gamma$ , by the corresponding change in structure of the boundary layer, so that the Chaplygin-Zhukovskiy postulate is fulfilled. In this case, the corresponding change of the boundary layer is quite small, so that it does not cause its breakdown. At the extreme points of the oscillation, because of the abrupt change of flow past conditions (angle of attack), such compensation becomes impossible, and the boundary layer breaks down, shedding the departing vortices.

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From this point of view, in oscillations of the wing, the boundary layer does not break down continuously, shedding more and more new vortices and forming a vortex sheet. Boundary layer breakdown occurs by individual drops, quanta, at the upper and lower points. In the

transition of the wing from one extreme point to the other, swelling of these drops is generated in the boundary layer. With sufficient flap velocity, they break away at the extreme points of the oscillation.

Consequently, the boundary layer, i.e., the viscosity of the fluid, is a unique regulator, which permits simultaneous fulfillment of the Thomson theorem and the Chaplygin-Zhukovskiy postulate.

It is completely clear that, if we knew exactly the conditions under which boundary layer equilibrium is preserved, we could exactly specify when the vortices are separated from it and, consequently, we would not have to make any supplementary physical hypotheses, in particular, the Chaplygin-Zhukovskiy postulate would not be needed. Unfortunately, despite the tremendous amount of work on the boundary layer, we know almost nothing of the conditions of its stability.

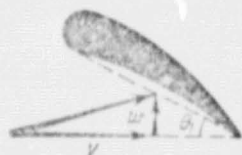


Fig. 2.

The hypothesis we have adopted permits determination of all elements of the resulting vortex street behind the wing. Actually, let  $\theta_1$  and  $\theta_2$  be the angles of

attack in lowering and raising the wing and  $\Gamma_1$  and  $\Gamma_2$ ,

correspondingly, be the circulation of the wing when lowering and raising it. Then, noting that, in lowering the wing, the relative velocity  $w$  is directed upward and gives an increase in angle of attack of  $\arctg \frac{w}{V}$

(Fig. 2), from the known formula, we obtain an expression for circulation in the form

$$\Gamma_1 = -\pi b \sqrt{V^2 + w^2} \sin\left(\frac{\alpha}{2} + \theta_1 + \arctg \frac{w}{V}\right).$$

In exactly the same way, in raising the wing, when the relative velocity is directed downward, we obtain a decrease in the angle of attack by  $\arctg \frac{w}{V}$  (Fig. 3) and, for the circulation, we obtain the expression

$$\Gamma_2 = -\pi b \sqrt{V^2 + w^2} \sin\left(\frac{\alpha}{2} + \theta_2 - \arctg \frac{w}{V}\right).$$

where  $\alpha$  is the camber of the wing skeleton.

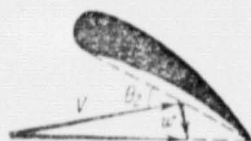


Fig. 3.

The resulting formulas are valid, if the effect of the departing vortices is not taken into account but, because of the physical hypothesis adopted above, they will be valid in the presence of vortices since, according to the hypothesis adopted, circulation  $\Gamma$  does not change, if lowering or raising of the wing continues without restriction, when the effect of the vortex street can be disregarded because of its remoteness.

It follows from this that, at the upper point of the oscillation, in the transition from raising to lowering, the circulation around the wing changes by the amount  $(\Gamma_1 - \Gamma_2)$ . Since, because of the Thomson

theorem, the circulation over the contour which envelops the wing does not change, at the upper point of the oscillation, the following vortex is shed

$$\gamma = -(\Gamma_1 - \Gamma_2).$$

On the assumption that  $\frac{\pi}{2} + \delta = \alpha$  and  $\frac{\beta_1}{2} = \alpha$ ,  $\frac{\beta_2}{2} = \delta$ , from the formulas written, we find

$$\gamma = 2\pi b \cos \alpha [V \sin \delta + \omega \cos \delta]. \quad (4)$$

In quite the same manner, we find that, at the lower points of the oscillation, vortices are shed from the wing with circulation  $-\gamma$

Thus, a double vortex street is generated behind the flapping wing, with the direction of rotation of the vortices shown in Fig. 4. The width of the street can be considered to equal the amplitude of the oscillations of the wing, because of the hypothesis made above.

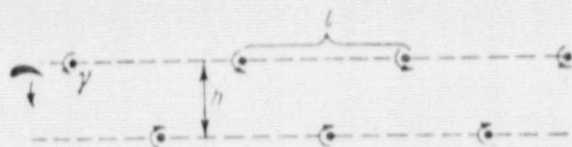


Fig. 4.

Let the period of oscillation of the wing be  $T$ . Let  $u_0$  be the velocity

added by the vortices of the street far from the wing by all the other vortices of the street. Then, far from the wing, we have (Fig. 4)

$$l = (V + u_0) T, \quad (5)$$

$$2h = \omega T. \quad (6)$$

Since it is known from vortex theory that

$$u_0 = \frac{\gamma}{2l} \operatorname{th} \frac{h\pi}{l}, \quad (7)$$

in the present case, by substituting the value of  $\gamma$ , we have

$$u_0 = \pi \frac{b}{l} \cos \alpha (V \sin \delta + \omega \cos \delta) \operatorname{th} \frac{h\pi}{l}. \quad (8)$$

From Eq. (5) and (6), we have

$$\frac{h}{l} = \frac{\frac{\omega}{V} - 2 \frac{h}{l}}{2 \frac{h}{l} \left[ \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} \right] \cos \alpha \left[ \sin \delta + \frac{\omega}{V} \cos \delta \right]}, \quad (9)$$



or, by using the values for the case of stability  $\frac{h}{l} = 0.281$  and  $\frac{h}{l} = \frac{1}{2}$ , we find

$$\frac{h}{l} = \frac{\frac{w}{V} - 0.562}{0.562 \left( 1 - \frac{w}{V} \cos \theta \right)} \quad (10)$$

From the equations obtained, it follows that

$$\frac{w}{V} > 0.562.$$

The preceding shows that the problem of calculation of the thrust of a flapping wing proves to be simpler than the problem of determination of the turbulent drag solved by Karman since, in the problem of calculation of the turbulent drag, there are no theoretical ways to determine the circulation of the departing vortices. /460

The physical scheme of the phenomenon assumed here has the following advantages over the scheme of the continuous vortex sheet (velocity discontinuity surface).

First, the sufficiently developed theory of vortex streets permits study of the question of their stability. Numerous theoretical studies lead to the conclusion that the stability conditions given by Karman correspond to vortex formations which, in any case, are more stable than all other vortex configurations of a similar type. Numerous experimental data confirm the same thing (Benard et al). For actually existing vortex systems, these data gave quite exact agreement with the theory of Karman. The existence of vortex streets behind a flapping wing was confirmed experimentally in the laboratories of Moscow State University.

Second, the scheme obtained here obviously is applicable to different types of change of velocity  $w$ , so that the very law of oscillations can take diverse forms.

Third, this method permits complete investigation of the dynamic aspect of the problem: determination of the forces which act on the flapping wing. We now proceed to this problem.

As is known, in the theory of turbulent drag, a similar problem was solved by Karman, by the use of the theorem of momentum (Euler theorem). Perhaps, it does not interfere to dwell somewhat on the questions of methods which are suitable for the solution of dynamic problems in theories of nonstationary motion.

There are two methods of solution of dynamic problems in the theory of wings. One of them is based on the use of the general theorems of dynamics (theorems of momentum) and the other, on direct calculation of the pressure on a unit surface of the body. This method, which is extremely convenient in the simplest cases, was used systematically by S.A. Chaplygin, who derived the first general equations of the theory /461



of wings, in the case of nonstationary motion.

There is no doubt that the method based on the general theorems of mechanics must be absolutely preferred in all problems which concern flows with the shedding of given vortex formations (dividing lines, streets, etc.). The presence of vortices moving around the wing makes impossible the direct determination of velocities and pressures on the wing surface. To a considerable extent, this is explained by the extremely slight progress in theories of nonstationary motion. The method based on the theorem of the change in momentum requires knowledge of the flow at some distance from the wing, where the flow always acquires a more regular form and levels out those characteristics which the wing has in nonstationary motion.

To the point, we note that, by the method of calculation of the pressure on a unit surface, the pressure cannot be calculated, even on a flat plate, under conditions of smooth flow past it.

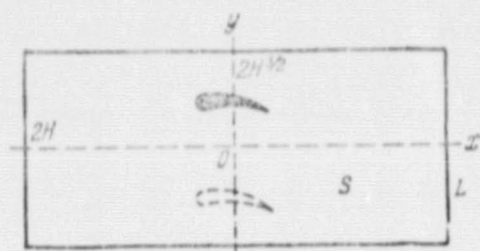


Fig. 5.

In the case of the theory of a flapping wing, the basic dynamic equations take the following form. The direct application of the theorem of momentum to the air mass enclosed in a control space, plotted in rectangle L (Fig. 5), leads to the equations

$$\begin{aligned} -X - \int_L p dy &= \frac{d}{dt} \int_S \rho u dx dy, \\ -Y + \int_L p dx &= \frac{d}{dt} \int_S \rho v dx dy, \end{aligned}$$

where X, Y are the components of the force acting on the wing and S is the area enclosed between outline L and the wing profile. In this case, the derivative with respect to t in the second part is the substantial derivative, since it does not concern the geometric outline, but the mass of fluid enclosed in it. With this taken into account and by noting that an amount of fluid  $\rho(u dy - v dx) dt$  flows through an element of outline L in time dt, we reduce the equations to the form

$$\begin{aligned} -X dt - \int_L p dy dt &= d \int_S \rho u dx dy + dt \int_L (u dy - v dx) \rho u, \\ -Y dt + \int_L p dx dt &= d \int_S \rho v dx dy + dt \int_L (u dy - v dx) \rho v, \end{aligned}$$

and, from this, after conventional transformations and integration with respect to t within the period, the following basic equation is obtained

$$\begin{aligned} (Y_0 + i X_0) T &= -\frac{\rho}{2} \int_0^T dt \int_L \left( \frac{dw}{dz} \right)^2 dz - \rho \int_0^T \int_L |\dot{z}|_0^T (dx - i dy) - \\ &\quad - \rho \left[ \int_S \int_0^T \frac{dw}{dt} dx dy \right]_0^T, \end{aligned} \quad (11)$$

where  $\phi$  is the velocity potential and  $X_0$ ,  $Y_0$  are the mean values of the components of the force in the period. The problem of determination of the forces acting on a wing in any periodic flow is reduced to the solution of this basic equation.

This problem has been the subject of study by a number of mathematicians. Some of these studies should be dwelt on, although nearly all of them do not concern the case of a flapping wing, but the case of turbulent drag, where the matter is somewhat simpler. The conclusion of a number of authors, basically repeating Karman, consists of the following.

If there were no formation of vortices within the control rectangle, it would be necessary to use the Chaplygin-Blasius equation to express the forces. If we take a rectangle of sufficiently large dimensions, moving with the vortex street, the motion on its surface can be considered steady state and, consequently, the Lagrange formula, in the /463

form  $p + \frac{\rho V^2}{2} = C$ , can be used to calculate the pressure. On the other hand, because of the formation of vortex pairs in period  $T$ , the momentum within the control rectangle increases by the amount  $\rho \gamma h$ , directed against the flow. Therefore, to the forces for the components in the direction of the flow, obtained by the Chaplygin-Blasius formula, there also must be added the increment of momentum per unit time due to vortex formation, equal to

$$\frac{\rho \gamma h}{T} = \frac{\rho \gamma h}{l} (V - u_0).$$

From the point of view of the formula derived above, in the present case, we obtain the following.

If we take mobile outline  $L$ , moving with the velocity of the vortex street, and the dimensions of the outline are sufficiently large, with respect to these axes, the middle term of formula (11) is not included, since the Lagrange equation must be used in the form for steady state flow, i.e.,

$$p + \frac{\rho V^2}{2} = C$$

instead of

$$p + \rho \frac{\partial \phi}{\partial t} + \frac{\rho V^2}{2} = p_0,$$

from which the term containing  $\phi$  is dropped.

Thus, the formula takes the form

$$(Y_0 + i X_0) T = -\frac{\rho}{2} \int_0^T dt \int_L \left( \frac{d\omega}{dz} \right)^2 dz - \rho i \left[ \int_s^T \frac{d\omega}{dz} dx dy \right]_0^T. \quad (12)$$

Further, upon noting that the second integral is the change in momentum within the control space, which occurs because of the departure of two vortices from it in time  $T$ , we find that the components of both terms along the axes prove to be  $\left(\rho \frac{v^2}{2\pi l} - \rho \frac{\gamma}{l} u_0 h\right) T$  and  $\rho \gamma h$ , in which the second term is simply the x axis component of the momentum carried away /464 by the pair of vortices. Since  $T = \frac{l}{V - u_0}$ , from the resulting considerations, we obtain the Karman formula

$$X = \left( \rho \frac{v^2}{2\pi l} - \rho \frac{\gamma}{l} u_0 h \right) + \rho \gamma (V - u_0) \frac{h}{l},$$

or

$$X = \rho \gamma (V - 2u_0) \frac{h}{l} + \frac{\rho v^2}{2\pi l}. \quad (13)$$

The Karman formula is derived in approximately the same way by Karman himself, in the Fuchs-Hopf monograph [7] and in Golubev [8].

The danger of this kind of reasoning, not reinforced by detailed analysis of Eq. (11), is clear from the following erroneous reasoning of N.Ye. Zhukovskiy [2]. Since it appears at first glance that the change in momentum in the period within the control space of fluid is securely connected with the flow past the body, it is reduced to the departure of two vortices from it, which carry away momentum equal to  $\rho \gamma h$ , which corresponds to a change in momentum equal to

$$\frac{\rho \gamma h}{T} = \rho \gamma (V - u_0) \frac{h}{l},$$

Zhukovskiy assumes the drag equal to the resulting expression, i.e.,

$$X = \rho \gamma (V - u_0) \frac{h}{l}$$

instead of formula (13). The error of such reasoning is that the effect of additional velocity  $u_0$  in the band of the street, directed against

the flow, which produces an additional escape of fluid at the sides of the control rectangle parallel to the direction of the flow, with the corresponding change in momentum, is not taken into account.

An extremely thorough analysis of the phenomena which occur here was given in the course of N.Ye. Kochin and N.V. Roze [3] where, by painstaking study of the change in momentum, expression of the drag (13), given above, was obtained. Here, the authors did not use any general formulas similar to Eq. (11).

The erroneous conclusion of N.Ye. Zhukovskiy presented above shows /465 the danger of reasoning similar to the reasoning of Karman presented above. The unreliability of the resulting conclusions is clear, if only

if only from the fact that the vortices which leave the control space, of course, change expression  $\frac{dw}{dz}$  and, therefore, it is not clear whether it is still necessary to add the term  $\gamma \Gamma h$  to the expression  $\int_0^T dt \int_L \left( \frac{dw}{dz} \right)^2 dz$ ,

since the corresponding momentum could be taken into account in the first term.

The basic equation, presented above in form (11), apparently was first used by Synge [4], also in the problem of the calculation of turbulent drag. However, the method of Synge can be applied to the case of a flapping wing and, generally, to all cases of periodic motions of a wing with vortex formation behind it, while all the preceding methods evidently cannot be applied to the case of primary circulation around the wing different from zero, at least, if they are not subjected to very substantial transformation.

However, the conclusion of Synge contains a significant error. Synge uses a control rectangle, which is fixed relative to the body around which flow occurs, so that a pair of vortices of the street leaves its boundaries in the period. Synge considers that the expression

$$\left[ \int_S \int \frac{dw}{dz} dx dy \right]_0^T$$

because of the periodicity of the motion, equals zero. Consequently, the matter is reduced to calculation of the expressions  $\int_0^T dt \int_L \left( \frac{dw}{dz} \right)^2 dz$  and  $\int_L |\vec{\gamma}|_0^2 (dx - t dy)$ .

We decide on calculation of the second integral. By an extremely complicated and confusing calculation, Synge obtains the value of  $\gamma h$  for it. However, without any calculations, it is easy to show that this integral equals zero.

Actually, let

$$|\vec{\gamma}|_t^T = \vec{\gamma}(x, y, t + T) - \vec{\gamma}(x, y, t) - f(x, y, t). \quad (14)$$

From this,

$$\frac{\partial \vec{\gamma}(x, y, t + T)}{\partial x} - \frac{\partial \vec{\gamma}(x, y, t)}{\partial x} = \frac{\partial f}{\partial x},$$

but, because of the periodicity of the flow at point  $(x, y)$ , with moving coordinates,

$$u(x, y, t + T) - u(x, y, t) = 0$$

or

$$\frac{\partial \vec{\gamma}(x, y, t + T)}{\partial x} - \frac{\partial \vec{\gamma}(x, y, t)}{\partial x} = 0.$$



Thus,  $\frac{\partial \Gamma}{\partial x} = 0$  and, similarly,  $\frac{\partial \Gamma}{\partial y} = 0$ . Consequently,

$$|\gamma|_t^T = f(t).$$

In particular,

$$|\gamma|_0^T = f(0) = \text{const.}$$

and, therefore,

$$\int_L |\gamma|_0^T (dx - i dy) = 0.$$

Consequently, it flows from the reasoning of Synge that

$$(Y_0 + i X_0) T = - \frac{\pi}{2} \int_0^T dt \int_L \left( \frac{dw}{dz} \right)^2 dz,$$

i.e., in periodic motion, on the average, the Chaplygin-Blasius theorem remains valid during the period of oscillation and, therefore, in a proper calculation which repeats the calculation of Synge, instead of formula (13), we obtain its first term, so that the erroneous reasoning of Synge reverses the error of Zhukovskiy, in a certain sense.

It is highly likely that the error of Synge, besides incorrect calculation of the integral  $\int_L |\gamma|_0^T (dx - i dy)$ , further, is the incorrect assumption that  $\left[ \int_s \int \frac{dw}{dz} dx dy \right]_0^T = 0$ , as a consequence of which, the

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momentum removed by the pair of vortices leaving the control rectangle is not taken into account.

However, with a very small change in the method used by Synge, it is easy to obtain the formula for turbulent drag and the force in the case of a flapping wing.

In order to avoid the difficulties involved with allowing for the effect on the value of the integral of the singular points which pass through the control surface in period T, it is sufficient to use, not a fixed control rectangle, but one moving together with the vortex street.

Here, as it turns out, calculation of the integral  $\int_L |\gamma|_0^T (dx - i dy)$ , which does not equal zero in this case, is extremely simplified. Thus, we obtain the use of a calculation which, in a certain sense, connects the method of Synge and the method used by N.Ye. Kochin and N.V. Roze.

If a rectangle is taken, with sides  $2H^{3/2}$  in the direction of flow and  $2H$  perpendicular to it, calculation gives the following values for the integrals

$$\left. \begin{aligned} \int_L |\varphi|_0^2 (dx - l dy) &= 0 \left( \frac{1}{H^2} \right), \\ \left[ \int_a^l \int_0^{\frac{d\varphi}{dz}} dx dy \right]_0^T &= \gamma h = 0 \left( \frac{1}{H^2} \right), \\ \frac{\gamma}{2} \int_0^T dt \int_L \left( \frac{d\varphi}{dz} \right)^2 dz &= \gamma u_0 \Gamma, \\ &+ \frac{\gamma^2}{2\pi l} \left[ \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1 \right] + 0 \left( \frac{1}{H^2} \right), \end{aligned} \right\} \quad (15)$$

and, from this, by substitution in basic Eq. (11), we obtain

$$\begin{aligned} Y_0 + i X_0 &= \gamma u_0 \Gamma - i \frac{\gamma^2}{2\pi l} \left[ \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1 \right] - \\ &= \gamma \Gamma (V + u_0) - \frac{\gamma^2 h}{l} l + 0 \left( \frac{1}{H^2} \right), \end{aligned}$$

by proceeding to the limit as  $H \rightarrow \infty$ , we obtain

$$Y_0 + i X_0 = \gamma V \Gamma - i \left[ \frac{\gamma^2}{2\pi l} \left( \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1 \right) + \frac{\gamma^2 h}{l} (V + u_0) \right],$$

from which

$$\left. \begin{aligned} X_0 &= -\frac{\gamma^2}{2\pi l} \left( \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1 \right) - \frac{\gamma^2 h}{l} (V + u_0), \\ Y_0 &= -\gamma V \Gamma. \end{aligned} \right\} \quad (16)$$

We note that, in the absence of primary circulation around the body past which flow occurs

$$\int_L |\varphi|_0^2 (dx - l dy) = 0$$

because of the equation

$$\frac{\partial \varphi(x+l, y, t+T)}{\partial x} = \frac{\partial \varphi(x, y, t)}{\partial x}, \quad (17)$$

satisfied with fixed axes and, consequently,

$$\frac{\partial \varphi(x_1, y_1, t+T)}{\partial x} = \frac{\partial \varphi(x_1, y_1, t)}{\partial x}$$

with moving axes. This is consistent with Eq. (14).

Since

$$u_0 = \frac{\gamma}{2l} \operatorname{th} \frac{h\pi}{l}.$$

Eq. (16) can be further written in the form

$$X_0 = \frac{\rho l^2}{2\pi d} - \frac{\rho \gamma h}{l} (V + 2u_0). \quad (18)$$

This formula differs from the Karman formula by the variable sign of  $\gamma$  and  $u_0$ , as must be expected from the vortex street formula.

By substituting here, according to the stability conditions of the street,

$$\frac{h}{l} = 0.281, \quad \gamma = lV\sqrt{8}u_0,$$

we obtain

$$X_0 = -\rho V^2 \left[ 0.314 \left( \frac{u_0}{V} \right)^2 + 0.794 \left( \frac{u_0}{V} \right) \right] \quad (19)$$

similar to the Karman formula for drag

$$X_0 = \rho V^2 \left[ 0.794 \left( \frac{u_0}{V} \right) - 0.314 \left( \frac{u_0}{V} \right)^2 \right],$$

obtained from (19), with change in the sign of  $u_0$ .

For the case of thrust, it is more advantageous to write its value through structural and kinematic data. By assuming

$$\gamma = 2\pi b \cos \alpha (V \sin \delta + w \cos \delta)$$

and

$$u_0 = \frac{b}{h} \cos \alpha (V \sin \delta + w \cos \delta) \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l},$$

under the stability conditions, we obtain

$$X_0 = -1.76 \rho b V^2 \cos \alpha \left( \sin \delta + \frac{w}{V} \cos \delta \right) \times \\ \times \left[ 1 + 0.25 \frac{b}{h} \cos \alpha \left( \sin \delta + \frac{w}{V} \cos \delta \right) \right]. \quad (20)$$

Since

$$\frac{b}{h} = \frac{\frac{w}{V} - 0.562}{0.35 \cos \alpha \left( \sin \delta + \frac{w}{V} \cos \delta \right)},$$

finally,

$$X_0 = -0.94 \rho b V^2 \cos \alpha \left( \sin \delta + \frac{w}{V} \cos \delta \right) \left( 1 + 1.2 \frac{w}{V} \right). \quad (21)$$

Finally, formula (15) gives the lift

$$Y_0 = -\rho V \Gamma_0$$

which fully corresponds to the theorem of Zhukovskiy. In this case

$$\Gamma = -\pi b V \sin \alpha \left( \cos \alpha - \frac{w}{V} \sin \alpha \right). \quad (22)$$

We find consideration of this question with some remarks on the origin of thrust from the "suction" at the leading edge. /470

It is well known that the "suction forces" are forces required for satisfaction of the Zhukovskiy theorem. For example, if the "suction" on the plate is not taken into account, the effect of the flow during flow past is reduced to force R, which is not normal to the velocity of the flow far from the plate, but normal to the plate. The development of "suction" Q has the result that the resultant of forces R and Q gives the Zhukovskiy force P. Thus, the "suction" simply is the satisfaction of the Zhukovskiy theorem (Fig. 6).

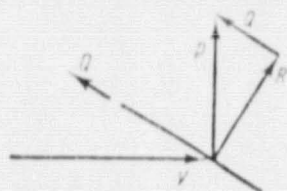


Fig. 6.

Since, in flaps of the wing, the direction of the resulting velocity changes during raising and lowering of the wing, thrust component of the Zhukovskiy force  $Q_1$  (Fig. 7) during lowering and drag  $Q_2$  (Fig. 8) during raising develop alternately.

If the mean thrust during an oscillation is  $Q = \frac{Q_1 - Q_2}{2}$ ,

we obtain the thrust expressed by the formula

$$Q = \pi \rho b \omega^2 \cos \theta.$$

If the value of Q is compared with the previously found expression of the force  $X_0$ , it turns out

that  $Q > X_0$ . This becomes clear, if it is noted that /471

the departing vortices increase the down wash when descending and increase it in rising.

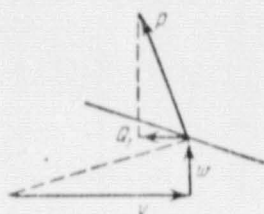


Fig. 7.

We note in conclusion that angles of attack  $\theta_1$  and  $\theta_2$  are limited by the critical angle of the profile.

Since the actual angles of attack during descent and ascent of the wing are, correspondingly,

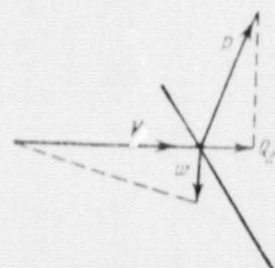


Fig. 8.

$$\theta_1 = \arctg \frac{w}{V} \quad \text{and} \quad \theta_2 = \arctg \frac{w}{V}.$$



It follows that the permissible angle of attack  $\theta_1$  when lowering, without separation of the flow, is less than angle  $\theta_2$  when raising. In this case, it also turns out that  $\theta_1 < 0$  when lowering. This possibly produces the illusion of backward scraping of the airflow by the wing.

If the free flight of a wing is considered, generally speaking, we find that the center of gravity of the flying system is raised during descent of the wing and is lowered during rising of the wing. From this, it is easy to show that the average lift should equal the weight of the flying system, if the flight is accomplished at the same altitude, on the average.

Further development of these considerations permits calculation of the maximum possible thrust. The following result is obtained here. Let  $\theta_1$  be the critical angle of attack of the wing, i.e., the maximum angle at which flow past is possible without stalling. Let  $\theta_0$  be the angle of attack at which, with flight speed  $V$ , the wing produces sufficient lift to keep the vehicle in the air. It then turns out that the maximum thrust is expressed by the formula

$$X_{\max} = -\pi \rho V^2 b (\theta_1 - \theta_0) 2 \frac{h}{l}.$$

We consider the difference between the greatest possible lift and the lift sufficient to keep the flight vehicle in the air

$$\pi \rho V^2 b \left( \frac{\alpha}{2} + \theta_1 \right) - \pi \rho V^2 b \left( \frac{\alpha}{2} + \theta_0 \right) = \pi \rho V^2 b (\theta_1 - \theta_0).$$

To a certain extent, this value is analogous to the buoyance margin of a ship. For an aircraft, it can be called the lift margin. It follows from the preceding that an aircraft wing can develop thrust, only in the event it has a lift margin. If there is no lift margin during flaps of the wing, the aircraft will lose altitude, i.e., drop downward. /472

All the preceding conclusions were reached, on the assumption of the existence of a vortex street behind the flapping wing. Consequently, these conclusions assume stability of vortex streets. It is easy to show that the stability conditions superimpose dependence of the oscillation period at given  $V$  and  $w$  on the dimension of the chord.

In fact, we saw that

$$\frac{w}{V} > 0.562.$$

On the other hand,

$$\frac{b}{h} = \frac{\frac{w}{V} - 0,502}{0,75 \cos z \left( \sin \delta + \frac{w}{V} \cos \delta \right)}$$

Consequently, from given  $w/V$ ,  $z = \frac{\pi}{2} + \frac{\delta_1 + \delta_2}{2}$  and  $\delta = \frac{\delta_1 - \delta_2}{2}$ , from  $b$ , we can find amplitude  $h$  and, then,  $\ell$ , from the equation

$$\frac{h}{\ell} = 0,281.$$

Since, on the other hand,

$$u_0 = \frac{b}{h} \cos z (V \sin \delta + w \cos \delta) \frac{h\pi}{\ell} \operatorname{th} \frac{h\pi}{\ell},$$

from given  $\operatorname{th} \frac{h\pi}{\ell} = \frac{1}{V^2}$  and  $h/\ell = 0,281$ , we find  $u_0$  and, consequently, we find period  $T$  (or the number of oscillations of the wing per second  $N$ ) by the formula

$$T = \frac{1}{N} = \frac{\ell}{V + u_0}.$$

As an example illustrating the conclusions obtained, we consider the following particular case. Let  $\alpha=0$ ,  $\delta=0$ ,  $\sigma=15^\circ$  and  $V=18$  km/hour= 5 m/sec. We then have the following results:

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	$b$	0,5 $\mu$	0,1 $\mu$	0,01 $\mu$
$\frac{w}{V} = 0,6$	$T$	1,8 $\frac{a}{\text{сек}}$	0,36 $\frac{a}{\text{сек}}$	0,036 $\frac{a}{\text{сек}}$
	$N$	0,55	2,8	28
$\frac{w}{V} = 0,7$	$T$	0,51 $\frac{a}{\text{сек}}$	0,1 $\frac{a}{\text{сек}}$	0,01 $\frac{a}{\text{сек}}$
	$N$	2	10	100
$\frac{w}{V} = 1$	$T$	0,17 $\frac{a}{\text{сек}}$	0,03 $\frac{a}{\text{сек}}$	0,003 $\frac{a}{\text{сек}}$
	$N$	6	33	333

Key: a. sec

Thus, for a wing of specific span and with given velocities  $w$  and

V, we obtain a completely determinate number of oscillations  $N$ , which increases with decrease in wing span. Whereas large wings flap comparatively slowly, the number of oscillations with small  $b$  is very large. The wing flaps of large birds are comparatively slow but, on the other hand, the rapid oscillation of the wings of insects (for example, dragonfly, etc.) evidently confirm these conclusions.

We finish discussion of questions connected with the generation of thrust of a flapping wing, with these general remarks on the flight mode.

The considerations developed above are far from limited to applications to theory of flapping wings. The development of the vortex theory of lift by N.Ye. Zhukovskiy and S.A. Chaplygin, and the remarkable works of N.Ye. Zhukovskiy on the vortex theory of propellers showed the completely exceptional importance of turbulent motion in nature and in various engineering applications. It can be thought that the highly particular form of vortex motion, more or less stable vortex formations in the form of vortex streets, has diverse applications. The happy thought of Karman, of subjecting dual vortex streets to systematic study from the point of view of their stability and to apply them to explanation of the development of drag in the flow of a fluid around a body, around which flow past is poor, made widely possible the application of similar considerations to a number of other mechanical phenomena, among which is the problem of determination of the thrust of a flapping wing. /474

In concluding my report, I wish to show that there are a number of phenomena in nature and in engineering, the mechanical explanation and theories of which apparently are most closely connected with the theory of vortex streets. Here, I deal with a number of problems associated with the generation of thrust in navigation.

The simplest example is the generation of thrust of a boat by rowing with oars. At first glance, the mechanical pattern of the phenomenon seems completely clear. In rowing, we engage a certain amount of water with the blade of the oar and, by driving it backwards, from the point of view of the theorem of momentum, we create the required backward momentum to generate thrust. However, more attentive examination of the phenomenon shows that it is not so simple at all. In fact, we have the generation of thrust by driving a certain amount of fluid backward, for example, in the case of the operation of a propeller, behind which a repelled flow of fluid forms. Moreover, in the case of operation of oars, we do not have any continuous stream driven backward. On the contrary, if we construct a control surface around the boat, vortices of liquid, the presence of which we always observe behind the boat, will leave through its boundaries. Each such pair of vortices carries momentum backwards, determined by the formula

$$Q = \rho \gamma h,$$

where  $\rho$  is the density of the liquid,  $\gamma$  is the vortex circulation,  $h$  is the distance between vortices and  $Q$  is the momentum carried away by the vortices. Mechanically, the pattern of vortex formation itself at the /475



end of the oar blade, because of the formation of a velocity discontinuity, is completely clear.

In the case of the generation of propulsive force of a boat by a stern oar, frequently used in fishing boats, we have a vortex pattern which still more closely approximates the vortex scheme, which we have in the generation of turbulent drag in the case of the thrust of a flapping wing, according to the theory reported above. Here, as a result of the action of the stern oar, a typical double vortex street is formed, with a checkerboard arrangement of vortices, with their direction of rotation opposite to that which occurs in Benard-Karman streets behind an obstacle past which flow is poor. From the point of view of thrust generation, the vortex scheme which results here completely coincides with that which we have in the case of a flapping wing.

Finally, the thrust generation mechanism in the work of the tail of a fish is extremely like the work of a stern oar. In the USSR, we have the research of V.V. Shuleykin [5], on the mechanism of generation of the propulsive force by fishes, constructed on completely different considerations, in which the undulating, sinuous movements of the fish body play an important part. From the point of view of the considerations developed here, the fish tail performs completely the same as the stern oar. The extremely complicated twisting motions of the tail, perhaps, are a very delicate mechanism, which forces the vortices to leave only with the greatest deflections of the tail. Of course, this is advantageous from the point of view of the turbulent nature of the generation of the propulsive force. The formation of quite intense vortices behind a swimming fish is in favor of such vortex considerations. Their presence has been determined experimentally, for example, in the photographs of Grey. In the theory proposed by V.V. Shuleykin, these vortices are considered a useless expenditure of energy on incidental phenomena while, from the point of view of the considerations developed here, these vortices, which carry away some of the momentum of the fluid, cause the generation of the propulsive force. It may be in favor of the theory developed here, that the head and body of the fish hardly oscillate during the work of the tail, similar to the hull of the boat during operation of the stern oar [6]. /476

It seems to me that all the preceding shows, to a sufficient extent, what great importance vortex motion has in explaining mechanical phenomena in nature and in engineering. Here, perhaps, it is pertinent to recall the words of the founder of vortex theory, Helmholtz, which N.Ye. Zhukovskiy put in an epigraph to his lectures on the theoretical foundations of aerial navigation: "So far as I can decide, at the present time, there is no basis for not using hydrodynamic equations for precise expression of the laws which control the motions of a fluid observed in reality."

Among the general hydrodynamics equations, the equations of vortex theory, in particular, the properties of vortex streets evidently are of exceptional importance. Yet, in theories of such phenomena, we still have a great many completely uninvestigated questions. Here, first and foremost, the conditions of vortex formation as a result of



boundary layer breakdown have not been explained. There is no doubt that this problem is most closely connected with the other, extremely complicated and completely unstudied problem of boundary layer stability.

In connection with the question of conditions of formation of vortex street, there is a second important question: of the possibility of the formation of streets which differ sharply from the direction of the general flow behind a body. In particular, theoretically, the formation of such streets is possible, if the flapping wing oscillates in such a manner that the rising time of the wing differs from the time of its descent. If such streets are physically possible, which perhaps only is solved experimentally, it is likely that, here, the explanation of that paradoxical phenomenon that insects and small birds can produce lift by the work of the wings, without moving forward, is hidden.

Finally, from the point of view of the application of vortex streets to phenomena observed in reality, there is great interest in developing theories of periodic vortex motions with indefinitely distributed turbulence, of which vortex streets are only a limiting, highly schematic case.

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## Questions of Theory of Flapping Wings<sup>1</sup>

One of the greatest achievements of N.Ye. Zhukovskiy, to whose memory the Lomonovskiy Lectures is dedicated, is his creation of the foundation of modern engineering aeromechanics: theory of wings and theory of propellers. The theory of wings was based on two fundamental ideas. First, the theory of polyvalent potentials in plane parallel flow, smoothly flowing past some contour; as is known, the use of the polyvalent potential led to the discovery of the basic Zhukovskiy theorem, on which calculation of the lift of a wing, thrust of a propeller, etc., is based. Second is the use of a carrier vortex scheme, which permits, with extreme success, an approximate solution of the problem of the work of a wing of finite span and construction of an approximate theory of propellers.

The theory of a wing of finite span is, of course, a highly particular case of the general theory of flow of a current past a body, i.e., of the classical problem which has attracted the attention of scientists since the time of Euler and D'Alembert, the founders of hydromechanics. Yet, despite the tremendous number of attempts to resolve the question of the forces with which a flow past a body acts on it, the theory of flow past a body is in a completely rudimentary state. If the classical studies of Stokes on the flow of a viscous fluid past a sphere, subsequently supplemented by Ozeyen, and the famous results of Euler and D'Alembert on the smooth flow of an ideal fluid past a body, which resulted in the famous Euler-D'Alembert paradox, we know practically nothing about the general case of flow past a body. There is no doubt that the reason for this is that we have no clear concept of the structure of the wake zone, in which the effect of features of the flow appear, which develop because of the presence in it of a body past which flow appears. Any construction of a theory of flow past a body begins with the construction of some physical scheme, which approximately represents the structure of the wake zone. In essence, we now have only two attempts to represent the structure of the wake zone.

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First, in general, accounting for the wake zone can be rejected, and it can be considered that it does not exist and that there is smooth flow past. This assumption leads to a clearly expressed contradiction to test data, to the Euler-D'Alembert paradox. Second, it can be considered that the wake zone is a region filled with a medium, which is fixed with respect to the body past which flow occurs, to which it is adjacent. This hypothesis leads to a theory of flow, which has been quite well developed for the case of plane parallel flow and is completely undeveloped for the case of three dimensional flow. It permits calculation of the drag, but the resulting value of the drag proves to be approximately half that actually observed. Besides, such a scheme evidently is completely inapplicable to the case of flow past wing shaped profiles at small angles of attack. The reason the theory of

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Paper at Lomonovskiy Lectures, 1947, dedicated to the 100th birthday of N.Ye. Zhukovskiy. Published in Uch. zap. MGU, issue 152, vol. 3, 1951 (1st ed.).

flow does not correspond to the test data is hidden in the fact that the basis of the hypothesis of the structure of the wake zone evidently does not correspond to reality at all, since the wake zone, as tests show, is not a zone of a fixed fluid, but a zone of a strongly turbulent fluid. Ozeyen and Burgers attempted to construct a scheme of the flow in the wake zone, with allowance for turbulent flow, but this attempt can not be considered successful, because of the artificiality of the hypotheses on which it is based and because of the disagreement of the resulting conclusions with test data. /479

Against this quite unreliable background, the theory of a finite wing is a remarkable achievement. The noteworthy success of this theory is because the scheme of the associated vortex and the system of free vortices connected with it permitted, from the physical point of view, a completely clear picture of the structure of the wake zone to be drawn, which undoubtedly is very close to what there actually is. Evidently, the clearly expressed vortex structure of the wake zone in the theory of wings forces the assumption that subsequent success in the theory of flow past a body can be achieved, only by means of a fortunate generalization of the vortex scheme of the structure of the boundary layer to the case of flow past a body of arbitrary shape.

A theory of a wing of finite span is the first significant success in the theory of the flow of a fluid past a body since the work of Stokes.

The second significant success is the idea proposed by Karman, on which the theory of vortex streets is based. The remarkable scheme of stable streets undoubtedly is progress in study of the structure of the wake zone. Unfortunately, we have not yet completely calculated the drag, since the circulation of the departing vortices remains unknown. Besides, there are great difficulties in transfer of the basic scheme of a vortex street to the case of flow of a fluid past an arbitrary body.

Probably, further success in solution of the basic problem of the hydromechanics of the flow past a body is connected with the question of study, on the one hand, of vortex formations of the vortex sheet type beyond a wing and, on the other hand, of vortex streets.

With the problem of determination of drag, the problem, in a certain sense the inverse, determination of the thrust of a flapping wing, is closely connected. /480

In some papers I have delivered at the USSR Academy of Sciences, I have shown how the theory of vortex streets can be applied to the calculation of thrust. A paper in press deals with the same problem.

Among the different cases of flight with flapping wings, absolutely remarkable from the hydrodynamic point of view is flight "in place" or, as biology calls it, "quivering flight" when birds or certain species of insects stay extremely steadily in one place by working the wings. Such flight can be observed, for example, by the dragonfly, some species of fly and small birds (hummingbirds, house martins, etc.).



If an explanation of the development of thrust and lift can be obtained, based on a scheme of departing vortex streets, and this explanation corresponds to reality, it must be expected that the same explanation can be applied to the case of flight in place. Such an explanation could significantly reinforce the explanation we gave earlier, of the development of thrust based on the theory of vortex streets.

We show that such an explanation can be obtained very simply, from the formulas we provided for the general case of flapping flight.

The following formulas for the lift and thrust of a flapping wing were derived in my work:

$$\text{lift } Y = \rho V \Gamma_0;$$

thrust

$$X = -0.95 \rho b V^2 \cos z \left( \sin z + \frac{w}{V} \cos z \right) \left( 1 + 1.2 \frac{w}{V} \right),$$

where  $b$  is the chord of the wing,  $w$  is the rate of rise and descent,

$$z = \frac{\mu}{2} + \frac{\theta_1 + \theta_2}{2}, \quad \dot{z} = \frac{\theta_1 - \theta_2}{2},$$

in which  $\theta_1$  and  $\theta_2$  are the angle of attack in lowering and raising the wing, respectively, and  $\mu$  is the angle which characterizes the camber of the wing.

We see from these formulas that, in the absence of forward velocity, i.e., with  $V=0$ , the lift reverts to zero ( $Y=0$ ), and the thrust retains a value different from zero, /481

$$X = -1.13 \rho b w^2 \cos z \cos \dot{z}, \quad (1)$$

Consequently, it is natural to look for an explanation of the development of lift in the absence of forward velocity in flying birds or insects converting thrust into lift, by means of change of the position of the body in space or by means of change in the nature of movement of the wings.

From the theory of flapping wings, we have the following expression for circulation in lowering and raising a wing

$$\Gamma_1 = -\pi b \sqrt{V^2 + w^2} \sin \left( \frac{\mu}{2} + \theta_1 + \arctg \frac{w}{V} \right)$$

and

$$\Gamma_2 = -\pi b \sqrt{V^2 + w^2} \sin \left( \frac{\mu}{2} + \theta_2 - \arctg \frac{w}{V} \right);$$

whence, from the assumption that  $V=0$  and that the wing is symmetrical (i.e.,  $\mu=0$ ), we obtain the following expressions for circulation

$$\Gamma_1 = -\pi h \omega \sin\left(\theta_1 + \frac{\pi}{2}\right), \quad (2)$$

$$\Gamma_2 = \pi h \omega \sin\left(\theta_2 - \frac{\pi}{2}\right). \quad (3)$$

Angles  $\theta_1$  and  $\theta_2$  are limited by the fact that the angle of inclination of the wing to the direction of the windstream, i.e., the effective angle of attack, cannot exceed the critical angle of attack of the profile. With all this taken into account, we obtain the following arrangement of the wing while descending and rising (Fig. 1).

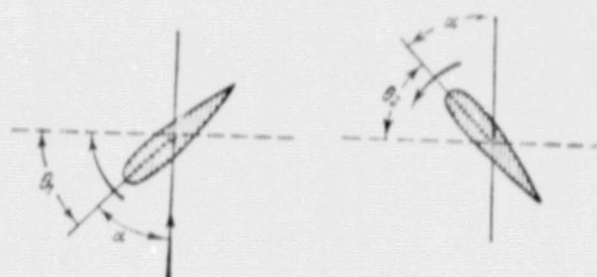


Fig. 1.

As is evident from Fig. 1, in descending, we have  $\Gamma_1 < 0$  and, in

rising,  $\Gamma_2 > 0$ . If the actual angle /482

of attack in a given case is designated by  $\alpha$ , in formulas (2) and (3), we will have

$$\theta_1 = -\frac{\pi}{2} - \alpha,$$

$$\theta_2 = \frac{\pi}{2} - \alpha,$$

and, therefore,

$$\Gamma_1 = -\pi h \omega \sin \alpha, \quad (4)$$

$$\Gamma_2 = \pi h \omega \sin \alpha. \quad (5)$$

As a result, for the circulation of the vortices of the departing stream, we obtain the following expressions:

for the vortices of the upper band

$$\gamma_1 = -(\Gamma_1 - \Gamma_2) = 2\pi h \omega \sin \alpha \quad (6)$$

and for the vortices of the lower band

$$\gamma_2 = -\gamma_1 = -2\pi h \omega \sin \alpha. \quad (6')$$

We now can calculate the thrust and all the elements which characterize the performance of a flapping wing in the condition  $V=0$  under consideration here.

Since we find from the formulas presented that,

$$\frac{\theta_1 + \theta_2}{2} = 0,$$

$$\frac{\theta_1 - \theta_2}{2} = -\left(\frac{\pi}{2} - \alpha\right)$$

and, consequently,  $\cos \delta = \sin \alpha$ , from formula (1), we obtain the following expression for the thrust

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$$X = 1,13; b \omega^2 \sin \alpha.$$

According to the foregoing, in order to obtain lift instead of thrust for flight in place, it is sufficient to turn the direction of the flaps to a right angle, and we obtain the arrangement shown in Fig. 2.



Fig. 2.

Key: a. Forward flap  
b. Backward flap

By introducing new axes, as shown in Fig. 2, we obtain the following values for lift Y and thrust X:

$$\left. \begin{aligned} X &= 0, \\ Y &= 1,13; b \omega^2 \sin \alpha. \end{aligned} \right\} \quad (7)$$

It is of interest to compare the resulting formula with the lift formula derived in forward flight, with the generation of a circulation force according to the theorem of N.Ye. Zhukovskiy.

We consider the lift which results in flight with velocity  $w$ , i.e., with a velocity equal to the rate of flapping of the wing, and with angle of attack  $\alpha$ . From the known formula for the lift in forward motion, we obtain the expression

$$Y_1 = \pi; b \omega^2 \sin \alpha, \quad (8)$$

and, therefore,

$$\frac{Y}{Y_1} = \frac{1,13}{\pi} = 0,35, \quad (9)$$

i.e., the lift in flight in place is 35% of the lift of forward flight with velocity  $w$ .

Thus, from the point of view of the amount of lift, flight in place proves to be not very efficient.

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The following interpretation can be given to this result. Since  $Y$  obviously is the lift required to keep the bird in the air, i.e., the lift required for flight, by designating the flight speed  $w_1$ , we can write

$$Y = \pi; b \omega_1^2 \sin \theta,$$

and, therefore, from (8) and (9),

$$\omega_1^2 : \omega^2 = 0,35,$$

i.e., we have  $w_1 = \sqrt{0,35} w$  or  $w_1 = 0,6 w$ .

Thus, in the case under consideration, the mechanism of flapping during flight in place is that the wing does not move upward and downward, but forward and backward, as shown in Fig. 3.

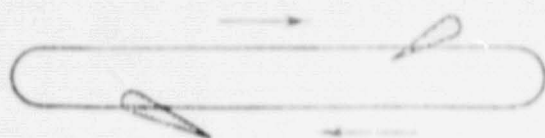


Fig. 3.

To achieve such wing movement, there must either be a mechanism which permits movement of the wings forward and backward during flapping, and not upward and downward, or the body of the flying bird or insect must be turned from the horizontal position to the

vertical. We note that the wing must rotate by an angle  $\pi - 2\alpha$  in flapping forward and backward. On the assumption that  $\alpha$  equals the critical angle, which is on the order of  $15^\circ$  for well streamlined wings, we find that the rotation of the angle around the horizontal axis reaches  $150^\circ$ .

With the value of  $\gamma$  known, we find the velocity of the street by the known formula

$$u_a = \frac{\gamma}{2l} th \frac{h\pi}{l}$$

In the case under consideration, by substituting the value of  $\gamma$  from Eq. (6) and assuming that the departing street satisfies the Karman stability condition  $th \frac{h\pi}{l} = \frac{1}{\sqrt{2}}$ , we obtain

$$u_a = \frac{\pi b}{\sqrt{2}l} \omega \sin \alpha. \quad (8')$$

Further, by designating the flap period  $T$  and the flap amplitude  $h$ , we obtain the equations

$$l = u_a T, \\ 2h = \omega T,$$

from which

$$\frac{l}{h} = \frac{2\pi b}{\sqrt{2}l} \sin \alpha = \frac{2\pi}{\sqrt{2}} \frac{h}{l} \frac{b}{h} \sin \alpha$$

and

$$\frac{b}{h} = \left(\frac{l}{h}\right)^2 \frac{1}{\pi \sqrt{2}} \frac{1}{\sin \alpha}$$

or, by assuming, according to the stability conditions,

$$\frac{b}{l} = 0.281,$$

$$\frac{b}{h} = \frac{1}{0.34 \sin \alpha},$$

from which

$$h = 0.34b \sin \alpha, \quad (9')$$

i.e., to fulfill the stability condition of the street, the flap amplitude, as Eq. (9') shows, must be, generally speaking, a small part of the chord of the wing. Thus, if it is considered that



$$\sin \alpha = \frac{1}{4},$$

then

$$h = 0.09b.$$

This small flap amplitude perhaps justifies the name, which is used in biological works for the characteristics of flight in place, "quivering flight."

Further,  $l$  can be determined by using the stability condition /486

$$l = \frac{h}{0.281} = 1.24b \sin \alpha, \quad (10)$$

and, by substituting the value of  $l$  in the expression for  $u_0$ , we obtain

$$u_0 = \frac{\pi}{1.24 \sqrt{2}} \omega = 1.8\omega \quad (11)$$

and, from the equation

$$l = u_0 T$$

we find flap duration  $T$

$$T = \frac{l}{u_0} = \frac{1.24b \sin \alpha}{1.8\omega} = \frac{\sin \alpha}{1.45} \frac{b}{\omega} \quad (12)$$

and, from this, for the number of oscillations per second, we have the expression

$$N = \frac{1}{T} = \frac{1.45}{\sin \alpha} \frac{\omega}{b}. \quad (13)$$

By using the expressions found, an expression for lift through the number of oscillations  $N$  can be found. Actually, from (13), we have

$$\omega = \frac{\sin \alpha}{1.45} N b = 0.7 N b \sin \alpha \quad (14)$$

and, by substituting the expression found in relationship (7), we obtain

$$Y = 0.57 \rho N^2 b^3 \sin^3 \alpha. \quad (15)$$

If both sides of Eq. (15) are multiplied by wing span  $L$  and it is noted that  $YL: bL = p$ , where  $p$  is the load on a unit area, we obtain

$$p = 0.57 \rho N^2 b^2 \sin^3 \alpha,$$

and, from this

$$N^2 = \frac{p}{0.57 \rho b^2 \sin^3 \alpha}.$$

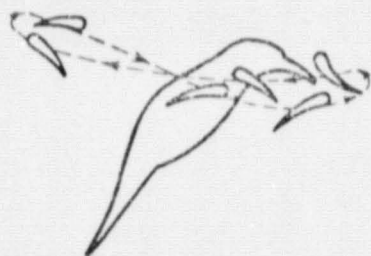
If the weight of a flying body is  $P$  and the area is  $S$ ,  $p=P/S$ , and the number of beats required to maintain the body in the air during flight in place is determined from the formula

$$N^2 = \frac{P}{0.07 \pi S^2 \rho} \quad (16)$$

Of course, it would be of extreme interest to verify the resulting /487 conclusions by test. However, there are no test data in this area. The only thing which could be used for verification, although roughly qualitative, is data on the flight of birds, of which there also are a completely inadequate quantity. Of course, it must be remembered that the application of the formulas derived for plane parallel flow in somewhat simplified flight conditions (abrupt change of velocity at the end points of the oscillation and rectilinear motion of the wing at constant speed in the first and second halves of the beat) to the case of the beating of bird wings which, undoubtedly, differ very sharply from the scheme examined here, cannot give a completely reliable test of the preceding theory, and it only can be a first, quite rough estimate.

From this point of view, the data on "quivering flight," presented in the dissertation of Gladkov Biologicheskiye osnovy poleta ptits [Biological Foundations of the Flight of Birds],<sup>2</sup> are of extremely great interest. Here, we present some data from this work on the flight of hummingbirds (*Melanotrochilus*).

Here are general flight characteristics: "when a hummingbird 'hangs' in the air in front of a flower, it executes rapid movements with the wings, almost in the horizontal plane. However, during this time, the body is in a nearly vertical position. Consequently (relative to the body of the bird), the general direction of wing motion is from the top down and slightly forward..." [Gladkov, p. 180].



"In this case, the wingtips describe an elongated figure eight (Fig. 4). In lowering the wing, and it is better to state, in moving /488 it forward, the wing becomes pronate and, in the reverse motion, it becomes supine. In this case, its bottom side proves to be on top, and it is most interesting that this is not simply an unusual orientation of the wing in space, but a temporary variation of the functions of a given side of the wings.

Fig. 4.

"In this case, the top side performs the function of the bottom and vice versa. In general, the same relationship results as that of an aircraft flying upside down ..."

"...Thus, as to the hummingbird, it can be stated that it regulates

<sup>2</sup>

Doctoral dissertation, Biological Department, Moscow State University, Feb. 1947.

the position of its body and wing motion, so that the thrust developed is directed upward and is equalized by the weight of the bird. We shall see below that quite the same forces develop in other birds, when they resort to quivering flight."

This general characterization is in complete agreement with the scheme of the flight of a flapping wing in place indicated above (Fig. 3). Data on the speed and number of beats are of great interest for a quantitative estimate. Here are these data.

"The linear velocity of the *Melanotrochilus* wing changes from 4 mm/sec (in the most extended forward position) to 20 mm/sec at the level of the body."

According to the determination of Stoltz and Zimmer (high speed camera), the number of beats is:

<i>Chlorostilbon</i>	36-39 per sec
<i>Melanotrochilus</i>	27-30 per sec

Stresemann and Zimmer, by means of stroboscopy, determined:

<i>Eupetomen macroura</i>	21-22 per sec
<i>Chlorestes coeruleus</i>	30-33 per sec
<i>Chrysolampis mosquitoes</i>	32-33 per sec
<i>Phaeternus ruber</i>	50-51 per sec

"In quivering flight, the seagull beats the wings still more often than during takeoff (6 times per second, during takeoff, 4-5 times, in normal flight, 3 times)."

A great difficulty in attempts to quantitatively compare the formulas obtained above with the data presented on the flight of birds is the lack of aerodynamic characteristics of birds, primarily the lack of load data. Therefore, loading data have to be extrapolated from available data. Of course, this is not very reliable. /489

There is the formula of Harting, which connects the wing area and weight of birds. This formula can be obtained from considerations of similarity, but it requires that the wing extension also be constant, as now follows from similarity considerations. Thus, the Harting formula is valid with uniform wing extension, which actually is not maintained.<sup>3</sup>

The Harting formula has the form

$$\frac{\sqrt{S}}{\sqrt{P}} = 4.2$$

(according to the definition of Marey), if  $S$  is measured in  $\text{cm}^2$  and

<sup>3</sup> See M.K. Tikhonravov, *Polet ptits i mashiny s mashushchimi kryl'yami* [The Flight of Birds and Vehicles with Flapping Wings], 1937, p. 16.



P in g. If the weight is in kg and the area in  $m^2$ , we obtain

$$\frac{\sqrt{S_1}}{\sqrt{P_1}} = 0.42,$$

from which

$$p = \frac{P_1}{S_1} = 5.7 \sqrt[3]{P_1}$$

i.e., the load decreases in proportion to the cube root of the weight of the bird. Apparently, the formula presented somewhat overstates the load for birds of low weight.

The weight of the hummingbird is 2 g. Thus, according to the Harting formula, the load is

$$p = 5.7 \sqrt[3]{2 \cdot 10^{-3}} = 0.57 \sqrt[3]{2}.$$

We can hardly make a large error by using a somewhat smaller load. We assume  $p=0.5$ . This can be achieved by a maximum increase of the wing surface.

With this assumption, we have

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$$\frac{P_1}{S_1} = \frac{1}{2}, S_1 = 2P_1.$$

By substituting the values found in formula (16), we obtain

$$N^2 = \frac{P}{0.57 \left( \frac{1}{8} \right) S_1 b^2 \sin^3 \alpha}.$$

By designating the extension of the wing by  $\lambda$ , we obtain  $b = \frac{L}{\lambda}$  and

$$\text{thus, } b^2 = \frac{bL}{L} = \frac{S_1}{L},$$

$$N^2 = \frac{P_1 L}{0.57 \left( \frac{1}{8} \right) S_1^2 \sin^3 \alpha} = \frac{2P_1 L}{0.57 P_1^2 \sin^3 \alpha},$$

i.e.,

$$N^2 = 3.5 \frac{L}{P_1 \sin^3 \alpha}.$$

For the hummingbird,  $P_1 = 2 \text{ g} = 2 \cdot 10^{-3} \text{ kg}$ ; consequently,

$$N = 42 \sqrt{\frac{L}{\sin^3 \alpha}}.$$

It can be considered that  $\lambda$  is from 1 to 2 and  $\alpha$  is from  $45^\circ$  to



30°. From this, we obtain

$$N_{\max} = 42 \sqrt{20} = 168,$$

$$N_{\min} = 42 \sqrt{2} = 71.$$

Apparently, the formulas derived above give a quite overstated number of oscillations. Perhaps, the effect of the wingtip shows up here. However, the order or magnitude is sufficiently close.

It follows from the formulas obtained that, with increase in weight  $P$ , the number of oscillations decreases. If this remark is applied to the seagull (grey-blue seagull, *Larus canus*,  $P=0.89$  kg,  $\lambda=7.6$ ), for the number of beats, we obtain an expression  $\sqrt{\frac{80}{18}} \approx 10$

times less than for the hummingbird. Consequently, according to the numbers presented above, for the seagull, the number of beats varies from 7 to 17. It was shown above that it is 6, which is sufficiently close to the theoretical value.

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## Studies on Theory of Flapping Wings<sup>1</sup>

### Introduction

The purpose of this work is to investigate the mechanism of thrust generation of a flapping wing. Despite the great number of studies of the unsteady motion of a wing, of the theory of an oscillating or flapping wing, in particular, the question of the causes of the generation of thrust remains far from explained.

If all kinds of mechanically unsubstantiated considerations, which lead to nothing, of the scraping off or repulsion backwards of air by the flapping of a wing are disregarded, in general, two basic points of view can be specified.

In a number of works, the generation of thrust of a flapping wing is attributed to the effect of suction forces, which form at the leading edge of thin wings. As is known, attention was drawn to the development of such forces in the work of N.Ye. Zhukovskiy [1]. In particular, N.Ye. Zhukovskiy attempted to explain the development of frontal drag in this way. This kind of consideration is extremely dubious as applied to wings, the profiles of which have a more or less rounded shape at the leading edge, and all the profiles normally used in aviation and bird wing profiles are such. Subsequently, we shall see that, from the point of view of general mechanical considerations, the generation of suction is essentially a manifestation of the development of circulation around the profile, and it follows from this that the effect of such suction on the generation of thrust only can be attributed to the change of circulation during the flapping of a wing and the shedding of vortices behind the wing which is connected with it.

Another point of view of the generation of thrust is connected with the concept of the formation of a velocity discontinuity line behind a flapping wing, with current lines around it distributed in a wave like manner [13]. The reasons for introduction of velocity discontinuity lines behind a wing are associated with the difficulties, in the case of a flapping wing with circulation changing during flapping, of tying in the Thomson theorem on the preservation of circulation, on the one hand, and the Chaplygin-Zhukovskiy postulate on the shedding of flow from a sharp trailing edge, on the other hand. However, the introduction of such discontinuity surfaces into the theory of a flapping wing complicates matters extremely. To a certain extent, the explanation of this is that small oscillations of a wing are considered almost exclusively in the theories. Moreover, the adoption of small oscillations of a wing, which is fully adequate in vibration theory, is completely unsuitable for the general theory of a flapping wing. Besides, the physical scheme itself is extremely questionable, since such velocity discontinuity surfaces are unstable and, evidently, they never actually are observed. A system of vortices, which is similar to the Karman streets, is observed behind a flapping wing in a test.

<sup>1</sup>

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Of course, they can be considered as the result of breakdown of a continuous discontinuity surface on the wing itself.

Wing oscillations of any amplitude are considered in this work, under conditions of plane parallel flow. The first problem which arises here is to find ways which permit reconciliation of the theorem of preservation of vortices with the Chaplygin-Zhukovskiy postulate, in the case of rejection of the formation of a velocity discontinuity surface behind the wing. The resolution of this question is based on taking account of the effect of the boundary layer which, thus, is considered as a certain mechanism which regulates the shedding of the flow from the wing.

The physical scheme on which the present work is based is the following. During flapping of a wing, vortices are shed from the wing at the extreme points of the oscillations. The vortices form a double vortex street behind the wing. Detailed analysis of such a street shows that the direction of rotation of the vortices in it is the reverse of the direction of rotation of the vortices in Karman streets, which are produced during flow past an obstacle. These considerations permit the use, for determination of the lift and thrust of a flapping wing, of the theorem of momentum, which was applied by N.Ye. Zhukovskiy with such success to the general theories of wings and, by Karman, in the problem of determination of the drag of bodies around which flow is poor. /493

Some of the results we have obtained were reported previously, in a note published in 1942 [4] and in a paper to a scientific conference at the Red Army Military Academy in July 1944 [5]. The general physical considerations on which the theory under consideration is based were reported in a paper to the general assembly of the USSR Academy of Sciences on 17 October 1944 [6]. One particular case of the problem of a flapping wing was examined in a paper to Lomonosovskikh chteniyakh [Lomonosov Lectures], at Moscow State University in 1947 [9].

## 1. Boundary Layer of a Wing

As is known, from the point of view of boundary layer theory, the effect of the viscosity of a fluid during movement of a body in a medium with a low coefficient of viscosity appears only in the immediate vicinity of the body past which flow occurs, within the so called boundary layer.



Fig. 1.

Let AB be the boundary of a body (Fig. 1) and  $A_1B_1$  be the outer boundary of the boundary layer.

Beyond the limits of the boundary layer  $A_1B_1$ , the fluid can be considered ideal. Since a fluid adheres to a body at the boundary of a solid body and, consequently, the velocity of the fluid is zero, and the velocity equals the velocity of the flow past the body of an ideal fluid at the outer boundary of layer  $A_1B_1$ , the circulation of the velocity along outline MNPQ differs



from zero. Consequently, from the kinematic point of view, the boundary layer region can be considered a region filled with an ideal turbulent fluid.

We examine what effect the boundary layer vortices have on the flow past conditions. Since, subsequently, we will consider flow past a wing, by using the usual method in theories of wings, we represent the outer part of the wing profile conformally on the outer part of a round cylinder of radius  $R$  (Fig. 2). Here, we locate the axes in such a manner that the sharp trailing edge of the wing is represented on the surface of the round cylinder by point  $z=R$ . In this case, the turbulent boundary layer changes to a certain turbulent layer  $S$ , located around the cylinder. Let the turbulent density in this layer equal  $\sigma$ . Then, the characteristic function of the flow around the cylinder, which corresponds to the flow around the wing takes the form

$$W = Ve^{-i\theta} \left( z + \frac{R^2 e^{2i\theta}}{z} \right) + \frac{\Gamma}{2\pi i} \ln z + \frac{1}{2\pi i} \int_S \ln \frac{z - re^{i\varphi}}{z - \frac{R^2}{r} e^{-i\varphi}} dz, \quad (1)$$

where  $re^{i\varphi}$  is the affix of the point of the boundary layer with turbulent density  $\sigma$ ,  $\Gamma$  is the circulation around the cylinder,  $V$  is the flow velocity at infinity and  $\theta$  is the angle it forms with the effective axis.

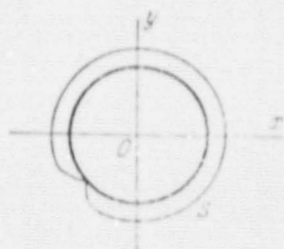


Fig. 2.

We require that the flow determined by characteristic function (1) satisfy the Chaplygin-Zhukovskiy postulate. As is known, to satisfy it, the flow velocity at point  $z=R$ , which corresponds to the trailing edge, must revert to zero. From this, we have the equation

$$\left( \frac{dW}{dz} \right)_{z=R} = 0,$$

or

$$Ve^{-i\theta} (1 - e^{2i\theta}) + \frac{\Gamma}{2\pi i R} - \frac{1}{2\pi i R} \int_S z \frac{r^2 - R^2}{r^2 - 2Rr \cos \varphi + R^2} dx dy = 0, \quad (2)$$

from which we obtain the circulation

$$\Gamma = -4\pi R V \sin \theta + \int_S z \frac{r^2 - R^2}{r^2 - 2Rr \cos \varphi + R^2} dx dy, \quad (3)$$

In the expression of characteristic function (1), the boundary layer vortices do not directly produce circulation around the wing,



since these vortices are paired at points  $re^{i\phi_1}$  and  $\frac{R^2}{r}e^{i\phi_1}$ , with the same intensity  $\sigma$ , but with the opposite direction of rotation. However, as equality (3) shows, these vortices affect the magnitude of the circulation indirectly, because of the necessity of satisfaction of the Chaplygin-Zhukovskiy postulate.

The effect of the boundary layer on vortex circulation, which is equivalent to the assumption  $\sigma=0$ , usually is not taken into account in theories of wings. Then, instead of Eq. (3), we obtain

$$\Gamma = -4\pi RV \sin \theta, \quad (4)$$

which is equivalent to the hypothesis

$$\iint_S z \frac{r^2 - R^2}{r^2 - 2Rr \cos \frac{\pi}{2} + R^2} dx dy = 0. \quad (5)$$

Condition (5) does not require the absence of boundary layer vortices. It only requires that the flow velocity due to these vortices at point  $z=R$ , which corresponds to the trailing edge of the wing on the cylinder, equal zero. It is possible that condition (5) reflects the circumstance that the boundary layer vortex system is unstable and that, actually, the boundary layer only can exist on condition that the expression

$$\left| \iint_S z \frac{r^2 - R^2}{r^2 - 2Rr \cos \frac{\pi}{2} + R^2} dx dy \right|$$

is sufficiently small. Otherwise, the boundary layer vortices are stripped from the wing.

## 2. Boundary Layer of a Wing in the Presence of a Vortex

We now apply the same considerations to the case when there is a vortex of intensity  $J$  near the wing. By changing to auxiliary plane ( $z$ ), for the characteristic function of the flow past the auxiliary round cylinder, we obtain the following expression /496

$$W = Ve^{-i\alpha} \left( z + \frac{R^2 e^{2i\alpha}}{z} \right) + \frac{\Gamma}{2\pi i} \ln z + \frac{J}{2\pi i} \ln \frac{z - r_1 e^{i\alpha}}{z - \frac{R^2}{r_1} e^{i\alpha}} + \frac{1}{2\pi i} \iint_S z \ln \frac{z - r e^{i\alpha}}{z - \frac{R^2}{r} e^{i\alpha}} dx dy, \quad (1)$$

where  $r_1 e^{i\alpha}$  is the affix of vortex  $J$ .

In this case, satisfaction of the Chaplygin-Zhukovskiy postulate

leads to the condition

$$\begin{aligned} Ve^{-i\alpha}(1 - e^{2i\alpha}) + \frac{\Gamma}{2\pi i R} + \frac{J}{2\pi i R K^2} \frac{K^2 - r_1^2}{2Rr_1 \cos \alpha + r_1^2} + \\ + \frac{1}{2\pi i} \int_S \int \frac{K^2 - r^2}{R(r^2 - 2Rr \cos \varphi + R^2)} dx dy = 0. \end{aligned} \quad (2)$$

We raise the following question: is it impossible to satisfy condition (2) with constant  $\Gamma$ ,  $J$ ,  $V$ ,  $\theta$  and arbitrary  $r_1$  and  $\alpha$ , i.e., with constant circulation around the wing, with permanent vortex  $J$ , with a velocity at infinity of fixed magnitude and direction, but with arbitrary position of vortex  $J$ ?

By assuming, in particular,  $r_1 = \infty$ , from Eq. (2), we obtain

$$\Gamma - J = -4\pi R V \sin \theta + \int_S \int \frac{r^2 - R^2}{r^2 - 2Rr \cos \varphi + R^2} dx dy, \quad (3)$$

where  $\sigma_\infty$  is the density of the boundary layer vortices at  $r_1 = \infty$ , i.e.,

when there is no external vortex. Evidently, in this case, the circulation around the wing is  $\Gamma - J$  and

$$\Gamma - J = -4\pi R V \sin \theta + \int_S \int \frac{r^2 - R^2}{r^2 - 2Rr \cos \varphi + R^2} dx dy. \quad (4)$$

Equality (4) is analogous to equality (3) of Section 1, for the case under consideration. On the other hand, from equality (2), we have

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$$\begin{aligned} \Gamma - J = -4\pi R V \sin \theta + J \frac{r_1^2 - R^2}{r_1^2 - 2Rr_1 \cos \alpha + R^2} + \\ + \int_S \int \frac{r^2 - R^2}{r^2 - 2Rr \cos \varphi + R^2} dx dy, \end{aligned} \quad (5)$$

which also can be written in the following manner

$$\begin{aligned} \Gamma - J = -4\pi R V \sin \theta + \int_S \int \frac{r^2 - R^2}{r^2 - 2Rr \cos \varphi + R^2} dx dy + \\ + J \frac{2R(r_1 \cos \alpha - R)}{r_1^2 - 2Rr_1 \cos \alpha + R^2} + \\ + \int_S \int (z - z_0) \frac{r^2 - R^2}{r^2 - 2Rr \cos \varphi + R^2} dx dy. \end{aligned}$$

From this we find that, if the condition

$$\int \frac{2R(r_1 \cos \alpha - R)}{r_1^2 - 2Rr_1 \cos \alpha + R^2} + \int_S \int \frac{r^2 - k^2}{r^2 - 2Rr \cos \varphi + k^2} dx dy = 0, \quad (6)$$

is satisfied, at any  $r_1$  and  $\alpha$ , i.e., with any position of vortex J, the Chaplygin-Zhukovskiy postulate is satisfied with constant  $\Gamma$ , J, V and  $\theta$ . Yet, condition (6), of course, can be satisfied by suitable selection of vortex density  $\sigma$ .

If it is assumed, as usually is done in theories of wings, that

$$\int_S \int \frac{r^2 - k^2}{r^2 - 2Rr \cos \varphi + k^2} dx dy = 0,$$

condition (6) takes the form

$$\int \frac{2R(r_1 \cos \alpha - R)}{r_1^2 - 2Rr_1 \cos \alpha + R^2} + \int_S \int \frac{r^2 - k^2}{r^2 - 2Rr \cos \varphi + k^2} dx dy = 0. \quad (6')$$

Equality (6) or (6') shows that the structure of the boundary layer depends on J,  $r_1$  and  $\alpha$ , i.e., on the intensity and position of

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the vortex external to the wing. The considerations stated show that the boundary layer of the wing can be considered a unique regulator of the shedding of the flow from the wing. The effect of the boundary layer has the result that, in the departure of vortices from the wing, to satisfy the Chaplygin-Zhukovskiy postulate, there is no necessity for the introduction of a continuous velocity discontinuity surface leaving the wing. For example, in the flapping of a wing, it can be considered that the vortices are shed from the wing only at the extreme positions of the wing. Subsequently, in the progressive departure of vortices shed from the wing, satisfaction of the Chaplygin-Zhukovskiy postulate is ensured by the regulating effect of the boundary layer.

In conclusion, we note that the term

$$\int_S \int \frac{r^2 - k^2}{r^2 - 2Rr \cos \varphi + k^2} dx dy$$

in Eq. (3) Section 1 is an allowance for the effect of the boundary layer on the magnitude of the circulation, i.e., accounting for the effect of air viscosity. The corresponding terms in the equations of Section 2 play a similar part. The adoption of condition (5) Section 1 or condition (6') Section 2 is equivalent to disregarding the effect of viscosity on the circulation.



### 3. Experimental Data

The physical considerations presented in this section are the basis of all subsequent theory. Therefore, there was great interest in their experimental verification. Such verification was performed in 1946 by Ya.Ye. Polonskiy [10].

A schematic representation of the apparatus he built for this purpose is given in Fig. 3. The basic part of the apparatus is the cart which rolls on rails along a water filled channel. A wing is attached to the cart which, during movement of the cart, performs oscillations perpendicular to the direction of movement of the cart. The cart and wing are moved by an electric motor. A camera, installed immovably above the trough, permits photography of the vortex formation which arises in the water behind the flapping wing. For their observation and photography, the water was sprinkled with a mixture of lycopodium powder and talc.

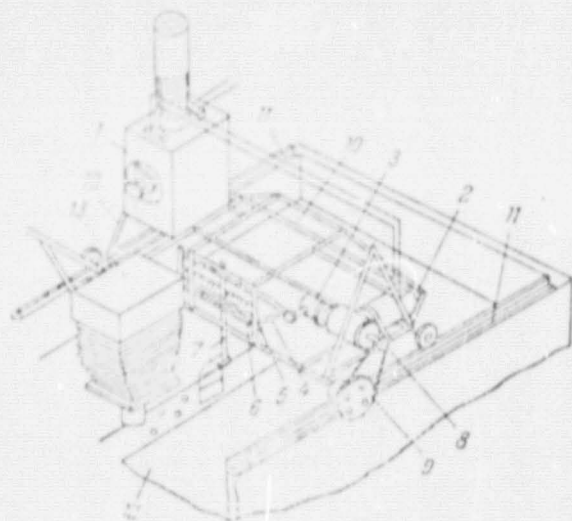


Fig. 3. Unit for experiment with flapping wing: 1. gas filled tube rectifier; 2. motor with radiator; 3. adjustable reducer with cam; 4. moving frame with roller; 5. spring; 6. movement guide frame; 7. holder with wing; 8. drive pulley; 9. cart drive pulley; 10. cart frame; 11. rails; 12. trough with guide lines; 13. camera.

A set of cams permitted various wing movement conditions to be obtained, for example, a change of wing flapping rate in a given direction during movement. Photographs obtained by Ya.Ye. Polonskiy are presented in Figs. 4, 5, 6 and 7. Here, Fig. 4 shows the "straight" street obtained when the speed of the wing was the same in both directions. Fig. 5, 6 and 7 give "skewed" streets, i.e., streets which deviate from the direction of motion of the cart. In the figures, this direction is indicated by the straight guidelines. The "skewed" streets result, when the rate of motion of the wings in each direction, i.e., its upward and downward flaps, differs.

The results of numerous tests led Ya.Ye. Polonskiy to the following conclusions.

"At the extreme positions of the wings, during the change from lowering to raising and from raising to lowering, a vortex is shed from the wing.

"The separation of the vortex in an intermediate position of the wing could not be obtained. In the transition of the wing from one extreme position to the other, a highly noticeable increase in turbulence of the fluid on the trailing edge of the wing occurs.





Fig. 4.



Fig. 5.



Fig. 6.

which change abruptly at the extreme points of the oscillations, further, remains constant during raising and lowering of the wing at  $w$  and  $-w$ , respectively. If the velocity of the flow at infinity is  $V$  and the angle of attack is  $\theta$ , in the absence of wing flapping, the circulation is determined by the known formula

$$\Gamma = Vb \sin\left(\frac{\pi}{2} + \theta\right).$$

"With uniform flapping of the wing (the rate of lowering equals the rate of raising), the vortices shed from it form a checkerboard type street. The axis of the street is parallel to the direction of forward movement of the wing (the trough axis). Relative to the stationary liquid, the street moves in the opposite direction to that of the forward movement of the wing. The direction of rotation of the vortex street is opposite to the direction of rotation of Karman street vortices.

The ratio of the width of the street  $h$  to the distance between vortices  $\lambda$  varied in different experiments, and it fluctuated between 0.287 and 0.47. The experimental value of quantity  $h/\lambda$  is generally the same for the street behind the flapping wing and a Karman street."

Thus, the experiment gives adequate confirmation of the physical considerations on which the theory developed in the present work is based.

#### 4. Determination of Circulation of Vortices Shed during Wing Flapping

We consider a wing with chord  $b$ , performing oscillations of amplitude  $h$  in plane parallel flow, and we will consider that the rate of rise of the wing and the rise of descent,

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Fig. 7.

that the vortices shed from the wing form only at the extreme positions of the wing, the effect of the departing vortices on the circulation during lowering or during raising of the wing need not be taken into account. Actually, it appears to us that a wing performing oscillations, from the time when it begins descending, stops flapping upward and downward, subsequently, continues endlessly to descend at velocity  $w$ . If the descent of the wing continues sufficiently long, it can be considered that the wing is in steady state motion, and its circulation  $\Gamma_0$  is determined by the formula

$$\Gamma_0 = \pi b \sqrt{V^2 + w^2} \sin\left(\frac{\pi}{2} + \theta - \arctg \frac{w}{V}\right). \quad (2)$$

Since we assume that the vortices shed from the wing form only at the extreme positions of the wing, after shedding of the vortex, when the wing was in the extreme upper position, there was no further shedding of vortices during lowering of the wing and, consequently, the circulation retained the value of (2) all during the descent of the wing. We note that, as was explained above, the effect of the departing vortices on the position of the stream shedding point is neutralized by changes in the structure of the boundary layer, i.e., in other words, by the effect of viscosity of the air, which ensures satisfaction of the Chaplygin-Zhukovskiy postulate.



Fig. 8.

Subsequently, we locate the axis as shown in Fig. 8. With this selection of the axes, the circulation around the wing proves to be negative. Further, if it is considered that the angle of attack relative to the direction of velocity  $V$  during lowering of the wing is  $\theta_1$ , finally, for the circulation

during lowering of the wing, we obtain the expression

$$\Gamma_1 = -\pi b \sqrt{V^2 + w^2} \sin\left(\frac{\pi}{2} + \theta_1 + \arctg \frac{w}{V}\right). \quad (3)$$

In quite the same manner, we find that, during raising of the wing, relative velocity  $w$  is directed downward, and the angle of attack is reduced by the amount  $\arctg w/V$ . With the angle of attack during raising of the wing designated  $\theta_2$ , for the circulation  $\Gamma_2$  during raising of the wing, we obtain the expression

$$\Gamma_2 = -2\pi b \sqrt{V^2 + w^2} \sin\left(\frac{\alpha}{2} + \theta_2 - \arctg \frac{w}{V}\right). \quad (4)$$

It follows from this that, in the transition from raising the wing to lowering it, the circulation changes by

$$\Gamma_1 - \Gamma_2.$$

Further, with  $\frac{\alpha}{2} + \theta$  designated  $\mu$ , we obtain

$$\Gamma_1 - \Gamma_2 = -2\pi b \sqrt{V^2 + w^2} \cos \frac{\mu_1 - \mu_2}{2} \sin\left(\frac{\mu_1 + \mu_2}{2} + \arctg \frac{w}{V}\right),$$

where

$$\mu_1 = \frac{\alpha}{2} + \theta_1 \text{ и } \mu_2 = \frac{\alpha}{2} + \theta_2;$$

let

$$\frac{\mu_1 + \mu_2}{2} = \lambda \text{ и } \frac{\mu_1 - \mu_2}{2} = \delta,$$

then

$$\Gamma_1 - \Gamma_2 = -2\pi b \sqrt{V^2 + w^2} \cos \delta \sin\left(\lambda + \arctg \frac{w}{V}\right),$$

or

$$\Gamma_1 - \Gamma_2 = -2\pi b \cos \delta [V \sin \lambda + w \cos \lambda]. \quad (5)$$

Since, according to the Thomson theorem, the circulation along the profile which envelops the wing should not change here, such a change in circulation around the wing should be accompanied by the shedding of a vortex of the same magnitude from the wing to the surrounding fluid, but in the opposite direction. With this vortex designated  $\gamma$ , we have

$$\gamma = 2\pi b \cos \delta (V \sin \lambda + w \cos \lambda). \quad (6)$$

In a quite similar manner, we show that, during the transition from lowering to raising the wing, vortex  $\gamma_1 = -\gamma$  is shed from it.

Thus, in flapping the wing, at the extreme points of the oscillation, vortices are shed from the wing, of intensity  $\gamma$  at the upper and  $-\gamma$  at its lower points and, therefore, a vortex street of the Karman street type forms behind the wing, with vortices of intensity  $\gamma$  arranged in checkerboard order. The direction of rotation of the vortices is the reverse of the direction of rotation in the streets formed behind barriers (Fig. 9).

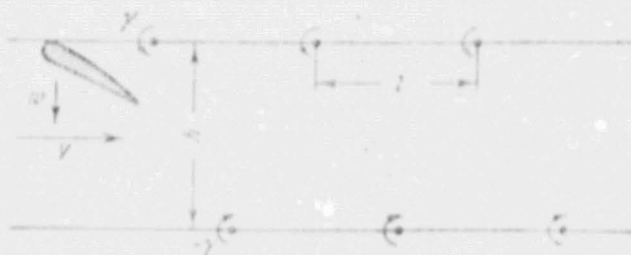


Fig. 9.

vortices of the street designated  $u_0$ , for the distance between two successive vortices of each row of the street  $l$  and for the width of the street  $h$ , we obtain the following expressions

$$\left. \begin{aligned} l &= (V + u_0)T, \\ 2h &= \omega T. \end{aligned} \right\} \quad (7)$$

It is known from the theory of vortex streets that

$$u_0 = \frac{\gamma}{2l} \operatorname{th} \frac{h\pi}{l}, \quad (8)$$

or, by substituting the expression for  $\gamma$  found above,

$$u_0 = \pi \frac{b}{l} \cos \varepsilon (V \sin \varepsilon + \omega \cos \varepsilon) \operatorname{th} \frac{h\pi}{l}.$$

Subsequently, it will be more convenient to have the ratio  $b/h$  instead of the ratio  $h/l$ . Obviously, the former is determined by the structure of the flapping wing. Therefore, we use  $u_0$  in the following form /506

$$u_0 = \frac{b}{h} \cos \varepsilon (V \sin \varepsilon + \omega \cos \varepsilon) \frac{\pi h}{l} \operatorname{th} \frac{\pi h}{l}. \quad (9)$$

By excluding oscillation period  $T$  from Eq. (7), we obtain the following relationship

$$\frac{b}{h} = \frac{\frac{\omega}{V} - 2 \frac{h}{l}}{2 \frac{h}{l} \left[ \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} \right] \cos \varepsilon \left[ \sin \varepsilon + \frac{\omega}{V} \cos \varepsilon \right]}. \quad (10)$$

Since  $b/h$  is positive, for the formation of the street, the form indicated above should satisfy the condition

$$\frac{\omega}{V} > 2 \frac{h}{l}. \quad (11)$$

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We apply Eq. (9) and (10) to the case when the vortex street formed behind the flapping wing satisfies the stability condition given by Karman. In this case, as is known,  $n/k=2.281$ ,  $\tan \frac{\pi}{2} = \frac{1}{\sqrt{2}}$ . By substituting these values in (10) and (11), we have

$$\frac{b}{h} = \frac{w}{V} \cos \left( \sin^{-1} \frac{1}{\sqrt{2}} \right) \quad (12)$$

and

$$\frac{w}{V} = 0.502 \quad (13)$$

With increase in  $w/V$ , as is evident from equality (12), ratio  $b/h$  increases, approaching  $\frac{1}{0.35 \cos 45^\circ} = \frac{2.86}{\cos 45^\circ}$ .

We consider the following simple example. Let  $\alpha=0$ ,  $\theta_1=\theta_2=0$ . This is the case of a flat plate, the inclination of which to the direction of forward movement during raising and lowering is the same. In this case,  $\delta=0$ ,  $d=0$  and, with various  $w/V$ , we have the following values of  $b/h$  (on the assumption that  $\theta$  is small and  $\cos \theta=1$ ):

$\frac{w}{V}$	0.7	0.8	0.9	1	2	3	4	5
$\frac{b}{h}$	1.8	1.2	0.95	0.8	0.68	0.63	0.61	0.55

## 5. Basic Equation of Theory of Flapping Wing

The basis of all subsequent studies is the equation which determines the strength of the action of the flow on a flapping wing. This equation expresses the theorem of momentum for a limited mass of air, which includes a flapping wing inside it.

We consider the  $(x_1, y_1)$  coordinate system, relative to which the wing performs periodic oscillations. Let  $O_1$ , the coordinate origin of this system, be the amplitude mean, which describes some point of the wing during oscillations. Let  $V$  be the velocity of the incident flow on the flapping wing far ahead of the wing. We direct the  $O_1 x_1$  axis along the direction of velocity  $V$ . According to the physical scheme of the phenomenon we have adopted, a double vortex street is shed from the wing, in which, far behind the wing, where the street can be considered completely formed, the vortices of the street move in the direction of the  $O_1 x_1$  axis at velocity  $V+u_0$ . We construct a second coordinate system

$(x, y)$ , relative to which the vortices of the street are stationary far from the wing. We assume that, at the initial time, the  $(x_1, y_1)$  and  $(x, y)$  axes coincide. Subsequently, the  $(x, y)$  axis is displaced relative to  $(x_1, y_1)$  along the direction of the  $O_1 x_1$  axis, with velocity  $V+u_0$ .

and that, in the period of the oscillation  $T$ , the origin  $O$  of the  $(x, y)$  coordinate system is displaced along the  $O_1 x_1$  axis by distance  $L=(V+u_0)T$ .

We stipulate that the  $(x_1, y_1)$  coordinate system is fixed and the

$(x, y)$  is moving. The air speed far from the wing and from the vortex street equals  $V-(V+u_0)=-u_0$  along the  $x$  axis.

We consider rectangle  $L$ , plotted on the moving  $(x, y)$  plane, with sides parallel to the coordinate axes and, we place the point of intersection of its diagonal at the fixed origin  $O$ . Let its sides be  $AB=2H$  and  $BC=2R$  (Fig. 10).

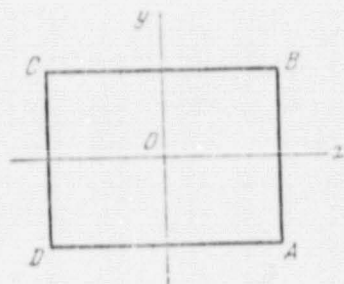


Fig. 10.

On  $ABCD$ , we plot rectangular parallelepiped  $Q$  with an altitude of one. On the assumption that the dimensions of the plotted rectangle are sufficiently large, so that the wing profile remains inside  $L$  during one oscillation period  $T$ , we apply the theorem of momentum to the mass of air included inside  $Q$ . With the components of the air pressure on the wing as  $X, Y$  and the air pressure as  $p$ , we write an equation which expresses the theorem of momentum in the form

$$\begin{aligned} X - \int_L p dy &= \frac{d}{dt} \int_S \rho u dx dy, \\ -Y + \int_L p dx &= \frac{d}{dt} \int_S \rho v dx dy, \end{aligned}$$

where  $S$  is the area occupied by the air inside  $L$  and the derivative with respect to  $t$  is understood to be substantial, with the flow of air through profile  $L$  taken into account.

By rewriting the equations in the form

$$\begin{aligned} -X dt + \int_L p dy dt &= d \int_S \rho u dx dy, \\ -Y dt + \int_L p dx dt &= d \int_S \rho v dx dy \end{aligned}$$

and noting that, in time  $dt$ , a quantity of air equal to  $\rho(u dy - v dx) dt$  flows out of an element of profile  $L$ , we can rewrite the equations in the form

$$\begin{aligned} -X dt + \int_L p dy dt &= d \int_S \rho u dx dy + dt \int_L (u dy - v dx) \rho u, \\ -Y dt + \int_L p dx dt &= d \int_S \rho v dx dy + dt \int_L (u dy - v dx) \rho v, \end{aligned}$$

in which partial differentials  $\partial$  refer to fixed profile  $l$ , which confines /509  
 3. By multiplying the first equation by  $l$  and adding with the second,  
 after insignificant transformations, we obtain

$$(Y + lX)dt = -dt \int_L \rho [(u dy - v dx)(ul + v) - p(dx - l dy)] - \\ - \partial \int_S \rho (v + lu) dx dy.$$

Since, according to the Lagrange theorem, in the case under consideration

$$p = p_0(t) - \frac{\rho}{2} \frac{d^2}{dt^2} - \frac{\rho}{2} (u^2 + v^2),$$

by substituting the value of  $p$  in the equation and noting that

$$\rho [(u dy - v dx)(ul + v) + \frac{1}{2} (u^2 + v^2)(dx - l dy)] = \\ = \frac{\rho}{2} (u - lv)^2 (dx + l dy) = \frac{\rho}{2} \left( \frac{dW}{dz} \right)^2 dz,$$

where  $W = \phi + i\psi$  is the complex flow potential, so that

$$\frac{dW}{dz} = u - lv,$$

after integration with respect to  $t$ , we obtain the equation in the form

$$(Y_0 + lX_0)T = -\frac{\rho}{2} \int_0^T dt \int_L \left( \frac{dW}{dz} \right)^2 dz - \\ - \rho \int_L \int_0^T [u dx - l dy] = \rho l \left[ \int_S \left( \frac{dW}{dz} \right)^2 dx dy \right]_0^T, \quad (1)$$

where  $X_0$ ,  $Y_0$  are the mean values of the components of the pressure of the air flow on the wing during an oscillation period. They are determined by the formulas

$$\int_0^T X dt = X_0 T, \quad \int_0^T Y dt = Y_0 T.$$

Thus, to find the values of  $X_0$  and  $Y_0$ , the values of the three in- /510  
 tegrals of the right side of equality (1) must be determined.



## 6. Determination of Function $W(z, t)$ and Profile $L$

Since, subsequently, we can increase the dimensions of profile  $L$  without limit, we find the analytical expression of function  $\frac{dw}{dz}$  in a region infinitely far from the point.

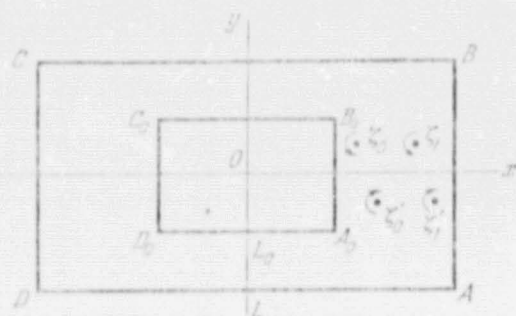


Fig. 11.

We plot profile  $L_0$ . Let its point of intersection of the diagonals be at the origin of moving coordinate system  $(x, y)$  and the sides parallel to the axes, and we make the dimensions of the sides so large that the vortex street outside of  $L_0$  can be considered completely formed

and to form a system of vortices fixed in plane  $(x, y)$ . Further, we will consider profile  $L_0$  to be constant. Besides, we

will consider the dimensions of profile  $L$  to be so large that  $L_0$  is inside  $L$ , and let  $\zeta_0, \zeta_1, \dots, \zeta_k$  ( $\zeta_k = \zeta_0 + kl$ ) be the affixes of the vortices of the upper row of the double vortex street and  $\zeta'_0, \zeta'_1, \zeta'_2, \dots, \zeta'_k$  ( $\zeta'_k = \zeta'_0 + kl$ ) be the affixes of the vortices of the lower row of the vortex street, which lie outside  $L_0$  (Fig. 11).

It is evident that the  $x$  axis (and  $x_1$  axis) always can be selected in such a way, that it passes in the middle of the vortex street. Then, with the locations shown in Fig. 11 and dimensions of  $L_0$  determined by 511 choice, we have

$$\zeta_k = \zeta'_k = \zeta_0 + kl = \frac{l}{2} + ih, \quad (1)$$

where  $h$  is the width and  $l$  is the distance between two successive vortices of one row of the street.

We formulate the expression

$$G(z) = \frac{1}{2\pi i} \sum_{k=-\infty}^{\infty} \left( \frac{1}{z - \zeta_k} - \frac{1}{z - \zeta'_k} \right). \quad (2)$$

It is evident that series (2) is absolute and converges uniformly outside of  $\zeta_k$  and  $\zeta'_k$ . Moreover, on the assumption that  $z = -m + ni$  ( $m > 0$ )



and  $\zeta_0 = i + \frac{1}{2}i$ , we obtain

$$\sum_{k=0}^N \left| \frac{1}{z - \zeta_k} - \frac{1}{z - \zeta'_k} \right| \leq |\zeta_k - \zeta'_k| \sum_{k=0}^N \frac{1}{(k! - m^2 - 1)^2 \left( \frac{b}{2} - a \right)^2} \frac{1}{(k! - m^2 - 1)^2 \left( \frac{b}{2} - a \right)^2} \\ \leq |\zeta_k - \zeta'_k| \sum_{k=0}^N \frac{1}{k! - m^2} \leq |\zeta_k - \zeta'_k| \left( \frac{1}{m^2} + \int_0^1 \frac{t^k}{k! - m^2} dt \right).$$

i.e.,

$$\left| \sum_{k=0}^N \left| \frac{1}{z - \zeta_k} - \frac{1}{z - \zeta'_k} \right| \right| \leq |\zeta_k - \zeta'_k| \left( \frac{1}{m^2} + \frac{1}{\ln 2} \right). \quad (3)$$

It follows from this that, as  $m \rightarrow \infty$ , the expression for  $G(z)$  tends towards zero.

We consider the difference

$$\frac{dW}{dz} - G(z) = f(z).$$

Outside of  $L_0$ , function  $f(z)$  does not have singular points at finite distance and, according to the preceding  $f(z)$ , with unrestricted increase,  $m$  tends toward  $-u_0$ . It follows from this that, in the region  $z = \infty$ , we have the expansion

$$\frac{dW}{dz} - G(z) = -u_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots \quad (4)$$

We note that  $\zeta_k, \zeta'_k$  are fixed in the  $(z)$  plane. By integrating both sides of equality (4) with respect to profile  $L$  and noting that, between  $L$  and  $L_0$ , function  $f(z)$  is holomorphic, and the number of points  $\zeta_k$  and  $\zeta'_k$  between  $L$  and  $L_0$ , because of selection of identical dimensions of profile  $L$  is the same, we have

$$2\pi i a_1 = \int_L \frac{dW}{dz} dz. \quad (4')$$

It is easy to find the mechanical meaning of  $a_1$ . At each moment, there is a certain number  $M$  of upper row and  $N$  of lower row vortices of the vortex street within  $L_0$ . Then, with the circulation around the wing profile designated  $\Gamma(t)$ , we obtain

$$\int_L \frac{dW}{dz} dz = \Gamma(t) - \gamma(M - N). \quad (5)$$

According to the hypotheses made, circulation  $\Gamma$  changes in steps, though  $\Gamma = \Gamma_1$  upon shedding of an upper row vortex, i.e., when lowering the wing and  $\Gamma = \Gamma_2$  in shedding a lower row vortex, i.e., in raising the wing. Moreover, because of the Thomson theorem, in the transition from lowering to raising the wing at the lower point of the oscillation, when a lower row vortex is shed, we have

$$\Gamma_1 = \Gamma_2 - \gamma. \quad (6)$$

Since the average circulation during an oscillation is determined by the equation

$$\Gamma = (\Gamma_1 \frac{T}{2} + \Gamma_2 \frac{T}{2}) : T = \frac{\Gamma_1 + \Gamma_2}{2},$$

according to (6), we have

$$\Gamma = \Gamma_1 + \frac{\gamma}{2} = \Gamma_2 - \frac{\gamma}{2}. \quad (7)$$

If we consider the wing lowering interval, we have the vortex arrangement shown in Fig. 12 and, therefore,  $M=N$  and, from equality (5), <sup>/513</sup>

$$\int_L \frac{dW}{dz} dz = \Gamma_1 = \Gamma - \frac{\gamma}{2}.$$

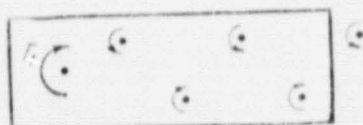


Fig. 12.

If we consider the interval which corresponds to raising the wing, we have the arrangement shown in Fig. 13 and, consequently,  $M-N=-1$  and, therefore,

$$\int_L \frac{dW}{dz} dz = \Gamma_2 - \gamma = \Gamma - \frac{\gamma}{2}.$$



Fig. 13.

Thus, the value of  $\int_L \frac{dW}{dz} dz$  does not depend

on time which, besides, follows directly from the Thomson theorem and, consequently, because of the selected dimension of  $L_0$ , from (4'), we have

$$2\pi ia_1 = -\frac{\gamma}{2} + \Gamma.$$

By substituting this value in expansion (4), we obtain the following analytical expression of  $\frac{dW}{dz}$  in the region of an infinitely remote point

$$\frac{dW}{dz} = -u_0 + \frac{\gamma}{2\pi i} \sum_{k=0}^{\infty} \left\{ \frac{1}{z - \zeta_k} - \frac{1}{z - \zeta'_k} \right\} + \frac{\gamma - \frac{\gamma}{2}}{2\pi i} \frac{1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \dots \quad (8)$$

Function  $\frac{dW}{dz}$  depends on  $z$  and on time  $t$  and, therefore, coefficients of expansion  $a_2, a_3$  are functions of  $t$ .

The  $(x, y)$  axis system is displaced relative to the  $(x_1, y_1)$  axes and, therefore, the location of the  $(z=x+iy)$  plane relative to the fixed coordinate system at moment  $t=0$  and  $t=T$  differs, as Fig. 14 shows. On the other hand, at any point A, immovably connected up to the fixed  $(x_1, y_1)$  axes, because of the periodicity of movement, the velocity at moment  $t=0$  and at moment  $t=T$  is the same. It follows from this that

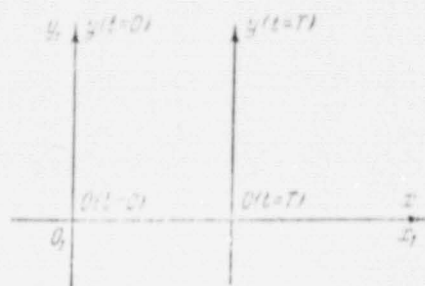


Fig. 14.

$$u - iv = \left( \frac{dW}{dz} \right)_{t=0} = \left( \frac{dW}{dz} \right)_{t=T}$$

and, by applying this condition to expansion (8), we have the equation

$$\begin{aligned} -u_0 + \frac{\gamma}{2\pi i} \sum_{k=0}^{\infty} \left\{ \frac{1}{z - \zeta_k} - \frac{1}{z - \zeta'_k} \right\} + \frac{\gamma - \frac{\gamma}{2}}{2\pi i} \frac{1}{z} + \frac{a_2(0)}{z^2} + \dots = \\ = -u_0 + \frac{\gamma}{2\pi i} \sum_{k=0}^{\infty} \left\{ \frac{1}{z - (\zeta_k + l)} - \frac{1}{z - (\zeta'_k + l)} \right\} + \\ + \frac{\gamma - \frac{\gamma}{2}}{2\pi i} \frac{1}{z - l} + \frac{a_2(T)}{(z - l)^2} + \dots \end{aligned}$$

from which, by carrying out contraction and expanding by  $1/z$  powers, we have the equations

$$0 = \frac{1}{z^2} \left[ -\frac{\gamma}{2\pi i} (\zeta_0 - \zeta'_0) + \frac{\gamma - \frac{\gamma}{2}}{2\pi i} l + a_2(T) - a_2(0) \right] + \frac{1}{z^3} (\dots) + \dots$$

from which

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$$[a_2]_0^T = a_2(T) - a_2(0) = \frac{1}{2\pi} \left[ -\left(\Gamma - \frac{\gamma}{2}\right)l + \gamma(z_0 - z'_0) \right],$$

and, since

$$z_0 - z'_0 = -\frac{l}{2} + hl,$$

finally,

$$[a_2]_0^T = \frac{\gamma h}{2\pi} + \frac{\Gamma}{2\pi} l. \quad (9)$$

We proceed to the selection of rectangle L. Subsequently, we will increase the dimensions of L without limit and, in this case, as inequality (3) shows, on side OD, the expression for G(z) satisfies the inequality

$$|G(z)| < \frac{\gamma}{2\pi} |z_0 - z'_0| \left\{ \frac{1}{R^2} + \frac{1}{lR} \frac{\pi}{2} \right\}, \quad (10)$$

and, therefore,

$$\left| \int_{cb} G(z) dz \right| < \frac{\gamma}{2\pi} |z_0 - z'_0| \left\{ 2 \frac{H}{R^2} + \frac{2}{l} \frac{H}{R} \frac{\pi}{2} \right\}. \quad (10')$$

With increase in the dimensions of L, expression (10') tends towards zero, if

$$\lim_{R} \frac{H}{R} = 0,$$

and, for this, it is sufficient to set

$$R = H^2. \quad (11)$$

Subsequently, we will use rectangle L with sides AB=2H and BD=2H<sup>3/2</sup>. With this selection of the side dimensions, expression  $\int_{cb} G(z) dz$  tends towards zero as M/H<sup>1/2</sup>, which we write down thus

$$\left| \int_{cb} G(z) dz \right| = O\left(\frac{1}{H^{1/2}}\right). \quad (12)$$

For later, we slightly refine the choice of rectangle L. We will draw side AB, so that it passes midway between two points  $\zeta_k$  and  $\zeta'_k$ ,

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as shown in Fig. 15. Since, evidently, side  $A_0 B_0$  can be drawn to distance  $l/4$  from  $\zeta_0$ ,  $m$ , the number of points  $\zeta_k$  and  $\zeta'_k$  between  $L_0$  and  $L$ , is determined from the equation

$$m = \frac{H^{3/4} - r}{l}, \quad (13)$$

where  $2r = B_0 C_0$ .

Let  $z = x + iy$  and  $x \leq H^{3/2}$ . It is easy to estimate the value of  $|G(z)|$ .

Actually, if  $\zeta_k = \xi_k + i\frac{h}{2}$  is set,

$$\left| \frac{\gamma}{2\pi l} \sum_0^m \left( \frac{1}{z - \zeta_k} - \frac{1}{z - \zeta'_k} \right) \right| \leq \frac{\gamma}{2\pi} |\zeta_0 - \zeta'_0| \sum_0^m \frac{1}{\sqrt{(x - \xi_k)^2 + \left(y - \frac{h}{2}\right)^2} \sqrt{(x - \xi'_k)^2 + \left(y + \frac{h}{2}\right)^2}},$$

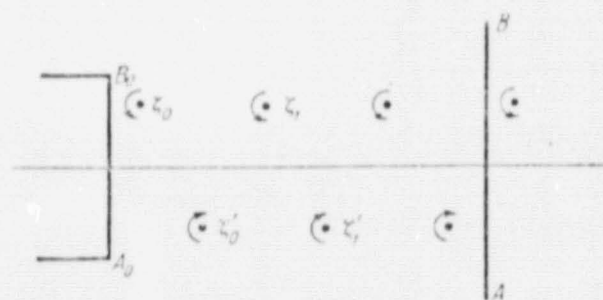
or

$$\left| \frac{\gamma}{2\pi l} \sum_0^m \left( \frac{1}{z - \zeta_k} - \frac{1}{z - \zeta'_k} \right) \right| < \frac{\gamma}{2\pi} |\zeta_0 - \zeta'_0| \frac{m}{\left(|y| - \frac{h}{2}\right)^2}.$$

On the other hand, for points outside  $L$ , we have

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$$|z - \zeta_k| = \sqrt{(x - \xi_k)^2 + \left(y - \frac{h}{2}\right)^2} > \sqrt{(k - m)^2 l^2 + \left(y - \frac{h}{2}\right)^2}$$



and, similarly,

$$|z - \zeta'_k| > \sqrt{(k - m)^2 l^2 + \left(y - \frac{h}{2}\right)^2},$$

and, therefore,

$$\begin{aligned} \left| \frac{\gamma}{2\pi l} \sum_{m+1}^{\infty} \left( \frac{1}{z - \zeta_k} - \frac{1}{z - \zeta'_k} \right) \right| &< \\ &< \frac{\gamma}{2\pi} |\zeta_0 - \zeta'_0| \sum_{m+1}^{\infty} \frac{1}{(k - m)^2 l^2 + \left(y - \frac{h}{2}\right)^2} < \\ &< \frac{\gamma}{2\pi} |\zeta_0 - \zeta'_0| \left\{ \frac{1}{\left(y - \frac{h}{2}\right)^2} + \int_0^{\infty} \frac{dx}{x^2 l^2 + \left(y - \frac{h}{2}\right)^2} \right\}. \end{aligned}$$

Fig. 15.

In this manner,

$$\begin{aligned} \left| \frac{\gamma}{2\pi l} \sum_0^{\infty} \left( \frac{1}{z - \zeta_k} - \frac{1}{z - \zeta'_k} \right) \right| &< \frac{\gamma}{2\pi} |\zeta_0 - \zeta'_0| \\ &- \zeta'_0 \left\{ \frac{H^{3/4} - r + l}{\left(|y| - \frac{h}{2}\right)^2} + \frac{\pi}{2l} \frac{1}{\left(|y| - \frac{h}{2}\right)} \right\}. \end{aligned} \quad (14)$$

Together with function  $G(z)$ , we examine another function  $G_0(z)$ , which is defined by the following equality

$$G_0(z) = \frac{\gamma}{2\pi i} \sum_{k=1}^n \left( \frac{1}{z - z_k} - \frac{1}{z - z'_k} \right). \quad (15)$$

We find some properties of this function for  $z$  on side AB or on its continuation, i.e., with  $z = H^{3/2} + iy$ . Since, in this case, evidently 518

$$\begin{aligned} |z - z_k| &= \sqrt{\left(\frac{l}{1+l}k - m\right)^2 + \left(y - \frac{h}{2}\right)^2} \\ &> \sqrt{l^2 k^2 - m^2 + \left(y - \frac{h}{2}\right)^2}, \\ |z - z'_k| &= \sqrt{\left(\frac{l}{1+l}k - m\right)^2 + \left(y - \frac{h}{2}\right)^2} \\ &> \sqrt{l^2 k^2 - m^2 + \left(y - \frac{h}{2}\right)^2}, \end{aligned}$$

then

$$\begin{aligned} |G_0(z)| &< \frac{\gamma}{2\pi} |z_0 - z'_0| 2 \left\{ \frac{16}{l^2} + \sum_{k=1}^n \frac{1}{k^2 l^2 + \left(y - \frac{h}{2}\right)^2} \right\} < \\ &< \frac{\gamma}{\pi} |z_0 - z'_0| \left\{ \frac{16}{l^2} + \frac{1}{l^2} + \int_1^\infty \frac{dx}{x^{3/2}} \right\}. \end{aligned}$$

i.e.,

$$|G_0(z)| < \frac{\gamma}{\pi} |z_0 - z'_0| \left\{ \frac{17}{l^2} + \frac{1}{2l^2} \right\} < \frac{\gamma}{\pi} |z_0 - z'_0| \frac{18}{l^2}. \quad (16)$$

Under the same conditions

$$|G(z)| < \frac{\gamma}{2\pi} |z_0 - z'_0| \sum_{k=1}^{\infty} \frac{1}{|z - l_k| \cdot |z - l'_k|} < \frac{\gamma}{\pi} |z_0 - z'_0| \frac{18}{l^2}. \quad (17)$$

Finally, on side AB

$$|G_0(z) - G(z)| = \frac{\gamma}{2\pi} |z_0 - z'_0| \sum_{k=0}^{\infty} \frac{1}{(H^{3/2} + r + kl)^2 + \left(|y| - \frac{h}{2}\right)^2}$$

or else

$$\begin{aligned} |G_0(z) - G(z)| &< \frac{\gamma}{2\pi} |z_0 - z'_0| \times \\ &\times \left\{ \frac{1}{H^{3/2} - r} + \int_0^\infty \frac{dx}{(H^{3/2} - r + lx)^2 + \left(|y| - \frac{h}{2}\right)^2} \right\}, \\ |G_0(z) - G(z)| &< \frac{\gamma}{2\pi} |z_0 - z'_0| \times \\ &\times \left\{ \frac{1}{H^{3/2} - r} + \frac{1}{l \left(|y| - \frac{h}{2}\right)} \operatorname{arctg} \frac{|y| - \frac{h}{2}}{H^{3/2} - r} \right\}. \end{aligned} \quad (18)$$

# 7. Finding Quantity $\int_L |\varphi|_a^T (dx - i dy)$

We have seen that

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$$\frac{dW}{dz} = u_0 + \frac{\gamma}{2\pi i} \sum_{k=1}^{\infty} \left( \frac{1}{z - \zeta_k} - \frac{1}{z - \zeta'_k} \right) + \frac{\Gamma - \frac{\gamma}{2}}{2\pi i} \frac{1}{z} + \frac{a_3}{2z^2} + \dots$$

Consequently,

$$W = -u_0 z + \frac{\gamma}{2\pi i} \sum_{k=1}^{\infty} \ln \frac{z - \zeta_k}{z - \zeta'_k} + \frac{\Gamma - \frac{\gamma}{2}}{2\pi i} \ln z - \frac{a_3}{z} - \frac{a_5}{2z^2} - \dots$$

and

$$\begin{aligned} \varphi = -u_0 x + \frac{\gamma}{2\pi} \sum_{k=1}^{\infty} [\arg(z - \zeta_k) - \arg(z - \zeta'_k)] + \\ + \frac{\Gamma - \frac{\gamma}{2}}{2\pi} \arg z - \operatorname{Re} \frac{a_3}{z} - \operatorname{Re} \frac{a_5}{2z^2} + \dots \end{aligned} \quad (1)$$

Upon noting that  $\zeta_k$ ,  $\zeta'_k$  and  $\Gamma$  do not depend on  $t$ , we have

$$[\varphi]_0^T = \varphi(x, y, T) - \varphi(x, y, 0) = -\operatorname{Re} \frac{[a_3]_0^T}{z} + \dots$$

and, therefore, according to formula (9), Section 6,

$$[\varphi]_0^T = \frac{\gamma h}{2\pi} \frac{x}{x^2 + y^2} - \frac{\Gamma}{2\pi} \frac{y}{x^2 + y^2} - \operatorname{Re} \frac{[a_3]_0^T}{2z^2} + \dots \quad (2)$$

We show that the integrals

$$\int_L \operatorname{Re} \frac{[a_k]_0^T (dx - i dy)}{z^{k-1}} \quad \text{with } k \geq 2$$

tends towards zero with unlimited increase of  $L$ . Actually,

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$$\left| \int_L \operatorname{Re} \frac{[a_k]_0^T (dx - i dy)}{z^{k-1}} \right| < \int_L \frac{|[a_k]_0^T| ds}{|z|^k} < 8 |a_k|_0^T \frac{H^k}{H^k}.$$

Consequently, from (2), we have

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<sup>1</sup> See (8), Section 6.

$$\int_L [\varphi]_0^T (dx - i dy) = \frac{\pi}{2} \int_L \frac{x(dx - i dy)}{x^2 + y^2} - \frac{H}{2} \int_L \frac{y(dx - i dy)}{x^2 + y^2} + O\left(\frac{1}{H^{1/2}}\right). \quad (3)$$

Since

$$\int_L \frac{x(dx - i dy)}{x^2 + y^2} = 2i \int_{-H}^H \frac{H^{1/2} dy}{H^2 + y^2} = 4i \operatorname{arctg} \frac{H}{H^{1/2}},$$

and, consequently, it tends towards zero as  $1/H^{1/2}$ , and

$$\int_L \frac{y(dx - i dy)}{x^2 + y^2} = 2 \int_{-H}^H \frac{H dx}{H^2 + x^2} = 4 \operatorname{arctg} \sqrt{H} = 4 \left( \frac{\pi}{2} - \operatorname{arctg} \frac{1}{H^{1/2}} \right),$$

by substituting all these values in equality (3), we obtain

$$\int_L [\varphi]_0^T (dx - i dy) = H + O\left(\frac{1}{H^{1/2}}\right). \quad (4)$$

#### 8. Calculation of Expression $\left[ \int_S \int \frac{dW}{dz} dx dy \right]_0^T$

Expression  $\left[ \int_S \int \frac{dW}{dz} dx dy \right]_0^T$  is the difference in value of integral

$\int_S \int \frac{dW}{dz} dx dy$ , calculated for time  $t=T$  and, consequently, for the

position of  $A_1 B_1 C_1 D_1$  of rectangle  $L$  relative to the fixed  $(x_1, y_1)$  axes (Fig. 16) and for the position of  $ABCD$  at time  $t=0$ . /521

We note that, because of the periodicity of the flow with respect to the  $(x_1, y_1)$  axes, the velocity at a point with the same coordinates at times  $t=0$  and  $t=T$  is the same and, consequently,  $\int_{A_1 B_1 C_1 D_1} \int \frac{dW}{dz} dx dy$



has the same value at times  $t=0$  and  $t=T$ , and it mutually cancels in the expression under consideration.

Thus,

$$\left[ \int_S \int \frac{dW}{dz} dx dy \right]_0^T = \int_{A_1 B_1 \bar{B}_1 A_1} (u - iv) dx dy - \int_{B_1 C_1 \bar{C}_1 B_1} (u - iv) dx dy, \quad (1)$$

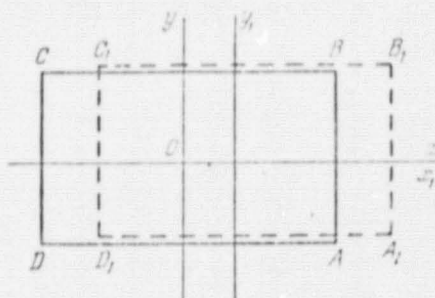
in which both integrals on the right side can be used for  $t=0$  and for  $t=T$ .

According to the results of Section 6 (relationships (10) and (12)), in region  $D_1 C_1 CD$  (Fig. 16)

$$|G(z)| < \frac{\gamma}{2\pi} |z_0 - z'_0| \left( \frac{1}{H^2} + \frac{\pi}{2l} \frac{1}{V H^2} \right)$$

and, consequently,

$$\left| \int_{D_1 C_1 CD} G(z) dx dy \right| < \frac{\gamma}{2\pi} |z_0 - z'_0| \left( \frac{2l}{H^2} + \frac{\pi}{H^2} \right). \quad (2)$$



On the other hand, in region  $A_1 B_1 BA$ , according to (18), Section 6,

$$|G_0(z) - G(z)| < \frac{\gamma}{2\pi} |z_0 - z'_0| \times \left\{ \frac{1}{H^2 - r} + \int_0^{\infty} \frac{dx}{(H^2 - r + lx) + \left( |y| - \frac{h}{2} \right)^2} \right\},$$

i.e.,

$$|G_0(z) - G(z)| < \frac{\gamma}{2\pi} |z_0 - z'_0| \left( \frac{1}{H^2 - r} + \frac{1}{l} \frac{1}{H^2 - r} \right),$$

and, since the area of  $A_1 B_1 BA$  equals  $2Hl$ ,

$$\left| \int_{A_1 B_1 \bar{B}_1 A_1} G_0(z) dx dy - \int_{A_1 B_1 \bar{B}_1 A_1} G(z) dx dy \right| < \frac{\gamma}{\pi} |z_0 - z'_0| (l+1) \frac{H}{H^2 - r}.$$

Further,

$$\left| \iint_{A_1 B_1 \bar{B} A} \frac{dx dy}{z^2} \right| < \iint_{A_1 B_1 \bar{B} A} \frac{dx dy}{x^2 + y^2} = \int_{H^2/4}^{H^2} dx \int_{-H}^H \frac{dy}{x^2 + y^2} = \\ = 2 \int_{H^2/4}^{H^2} \frac{dx}{x} \operatorname{arctg} \frac{H}{x} < \pi \ln \frac{H^2/4}{H^2/4 - 1} < \pi \frac{1}{H^2/4}.$$

Thus, to within terms on the order of  $1/H^{1/2}$ , in integration in region  $D_1 C_1 CD$ , the following can be set

$$\frac{dW}{dz} = -u_0 + \frac{\Gamma - \frac{\gamma}{2}}{2\pi l} \frac{1}{z} \quad (3)$$

and, in  $A_1 B_1 BA$ ,

$$\frac{dW}{dz} = -u_0 + \frac{\gamma}{2\pi l} \sum_{k=1}^{\infty} \left( \frac{1}{z - z_k} - \frac{1}{z - \bar{z}_k} \right) + \frac{\Gamma - \frac{\gamma}{2}}{2\pi l} \frac{1}{z}$$

or

$$\frac{dW}{dz} = -u_0 + \frac{\gamma}{2H} \left\{ \operatorname{ctg} (z - z_0) \frac{\pi}{l} - \operatorname{ctg} (z - \bar{z}_0) \frac{\pi}{l} \right\} + \\ + \frac{\Gamma - \frac{\gamma}{2}}{2\pi l} \frac{1}{z}. \quad (4)$$

We note further that rectangle  $A_1 B_1 BA$  is congruent with  $D_1 C_1 CD$ , and that point  $(x, y)$  in  $A_1 B_1 BA$  corresponds to point  $(-x-l, y)$  in  $D_1 C_1 CD$ , i.e., point  $z=x+iy$  in  $A_1 B_1 BA$  corresponds to point  $z_1=-x-l+iy$ , i.e.,  $z_1=-\bar{z}-l$  in  $D_1 C_1 CD$  ( $\bar{z}=x-iy$ ).

From this, we have

$$\left[ \iint_S \frac{dW}{dz} dx dy \right]_0^T = \frac{\Gamma - \frac{\gamma}{2}}{2\pi l} \iint_{A_1 B_1 \bar{B} A} \left[ \frac{1}{z} + \frac{1}{\bar{z}+l} \right] dx dy + \\ + \frac{\gamma}{2H} \iint_{A_1 B_1 \bar{B} A} \left[ \operatorname{ctg} (z - z_0) \frac{\pi}{l} - \operatorname{ctg} (z - \bar{z}_0) \frac{\pi}{l} \right] dx dy + O\left(\frac{1}{H^2}\right).$$

Yet

$$\frac{1}{z} + \frac{1}{\bar{z}+l} = \frac{1}{z} + \frac{1}{z} - \frac{l}{z^2} + \dots$$

and

$$\iint_{A_1 B_1 \bar{B} A} \left[ \frac{1}{z} + \frac{1}{\bar{z}+l} \right] dx dy = \iint_{A_1 B_1 \bar{B} A} \frac{2x}{x^2 + y^2} dx dy + O\left(\frac{1}{H}\right).$$

Since

$$\begin{aligned} \int_{A/B, H, 1} \int_{A/B, H, 1} \frac{2x}{x^2 + y^2} dx dy &= 2 \int_{H^2+1}^{H^2} dx \int_H^H \frac{x dy}{x^2 + y^2} = \\ &= 4 \int_{H^2+1}^{H^2} \arctg \frac{H}{x} dx < 4 \arctg \frac{H}{H^2+1} \end{aligned}$$

and, consequently,

$$\int_{A/B, H, 1} \left[ \frac{1}{z} + \frac{1}{\bar{z}-1} \right] dx dy = O\left(\frac{1}{H^2}\right).$$

Finally, we have

$$\begin{aligned} \left[ \int_{A/B, H, 1} \int_{A/B, H, 1} \frac{dW}{dz} dx dy \right]_{\alpha}^{\tau} = \\ = \frac{\gamma}{2H} \int_{A/B, H, 1} \int_{A/B, H, 1} \left[ \operatorname{ctg} (z - z_0) \frac{\pi}{l} - \operatorname{ctg} (z - z'_0) \frac{\pi}{l} \right] dx dy + O\left(\frac{1}{H^2}\right). \end{aligned} \quad (5)$$

We consider auxiliary flow in the  $(z)$  plane with complex potential  $W_1$ , which satisfies the equation

$$\frac{dW_1}{dz} = \frac{\gamma}{2H} \left[ \operatorname{ctg} (z - z_0) \frac{\pi}{l} - \operatorname{ctg} (z - z'_0) \frac{\pi}{l} \right]. \quad (6)$$

On the assumption that

$$W_1 = \varphi_1 + i\psi_1 = \frac{\gamma}{2\pi l} \ln \frac{\sin (z - z_0) \frac{\pi}{l}}{\sin (z - z'_0) \frac{\pi}{l}}, \quad (7)$$

we have

$$\psi_1 = \frac{\gamma}{2\pi l} \ln \left| \frac{\sin (z - z_0) \frac{\pi}{l}}{\sin (z - z'_0) \frac{\pi}{l}} \right| = \frac{\gamma}{4\pi l} \ln \frac{\sin (z - z_0) \frac{\pi}{l} \sin (\bar{z} - \bar{z}'_0) \frac{\pi}{l}}{\sin (z - z'_0) \frac{\pi}{l} \sin (\bar{z} - \bar{z}_0) \frac{\pi}{l}},$$

or

$$\psi_1 = -\frac{\gamma}{4\pi} \ln \frac{\cos [z + \bar{z} - (z_0 + \bar{z}'_0)] \frac{\pi}{l} - \cos [z - \bar{z} - (z_0 - \bar{z}'_0)] \frac{\pi}{l}}{\cos [z + \bar{z} - (z'_0 + \bar{z}_0)] \frac{\pi}{l} - \cos [z - \bar{z} - (z'_0 - \bar{z}_0)] \frac{\pi}{l}}.$$

Since, with

$$\begin{aligned} z = x + iy, \quad z_0 = \xi + i\eta, \quad z'_0 = \xi' - i\eta' \\ z + \bar{z} = 2x, \quad z - \bar{z} = 2iy \end{aligned}$$

etc.,

$$\phi_1 = -\frac{\gamma}{4\pi} \ln \frac{\cos 2l(y-\eta)\frac{\pi}{l} - \cos 2(x-\xi)\frac{\pi}{l}}{\cos 2l(y-\eta')\frac{\pi}{l} - \cos 2(x-\xi')\frac{\pi}{l}},$$

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or

$$\phi_1 = -\frac{\gamma}{4\pi} \ln \frac{e^{\frac{2iy-\eta}{l}\frac{\pi}{l}} + e^{-\frac{2iy-\eta}{l}\frac{\pi}{l}} - 2\cos 2(x-\xi)\frac{\pi}{l}}{e^{\frac{2iy-\eta'}{l}\frac{\pi}{l}} + e^{-\frac{2iy-\eta'}{l}\frac{\pi}{l}} - 2\cos 2(x-\xi')\frac{\pi}{l}}. \quad (8)$$

By using (6) and by assuming  $\frac{dW_1}{dz} = u_1 - iv_1$ , we obtain

$$\begin{aligned} \frac{\gamma}{2li} \int_{A,B,BA} \int \left[ \operatorname{ctg} (z - z_0) \frac{\pi}{l} - \operatorname{ctg} (z - z_0') \frac{\pi}{l} \right] dx dy = \\ = \int_{A,B,BA} \int \frac{\partial v_1}{\partial y} dx dy + i \int_{A,B,BA} \int \frac{\partial u_1}{\partial x} dx dy. \end{aligned}$$

From Eq. (7), we have

$$W_1(z+l) = W_1(z) + 2N\gamma,$$

where  $N$  is an integer. It follows from this that

$$\phi_1(x+l, y) = \phi_1(x, y).$$

Yet,

$$\int_{A,B,BA} \int \frac{\partial v_1}{\partial x} dx dy = \int_{-H}^H [\phi_1(H^2, y) - \phi_1(H^2 - l, y)] dy = 0$$

and

$$\int_{A,B,BA} \int \frac{\partial u_1}{\partial y} dx dy = \int_{H^2-l}^{H^2} [\phi_1(x, H) - \phi_1(x, -H)] dx, \quad (9)$$

and, at large  $H$ , from Eq. (8), we have

$$\begin{aligned} \phi_1(x, H) &= -\frac{\gamma}{4\pi} \ln e^{\frac{2(\eta-\eta')}{l}\frac{\pi}{l}} + \varepsilon = -\frac{\gamma}{2l}(\eta - \eta') + \varepsilon, \\ \phi_1(x, -H) &= -\frac{\gamma}{4\pi} \ln e^{\frac{2(\eta-\eta')}{l}\frac{\pi}{l}} + \varepsilon_1 = -\frac{\gamma}{2l}(\eta - \eta') + \varepsilon_1 \end{aligned}$$



and, therefore,

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$$\int_{A, B, C} \frac{\partial \psi_1}{\partial y} dx dy = \int_{H, C, 1} \frac{\gamma}{l} (\psi - \psi') dx + \varepsilon_2 = \\ = \gamma (\psi - \psi') + \varepsilon_2 = \gamma h + \varepsilon_2,$$

where  $\varepsilon$ ,  $\varepsilon_1$ ,  $\varepsilon_2$  tend toward zero as  $1/H$ .

From this, we have the final result

$$\left[ \int_{A, B, C} \frac{dW}{dz} dx dy \right]_0^T = \gamma h + O\left(\frac{1}{H}\right). \quad (10)$$

### 9. Calculation of Expression $\frac{\gamma}{2} \int_L \left( \frac{dW}{dz} \right)^2 dz$

As was shown above,

$$\frac{dW}{dz} = -u_0 + \frac{\gamma}{2\pi l} \sum_0^\infty \left( \frac{1}{z - \zeta_k} - \frac{1}{z - \zeta'_k} \right) + \frac{\Gamma - \frac{\gamma}{2}}{2\pi l} \frac{1}{z} + \frac{a_2}{z^2} + \dots = \quad (1)$$

$$= -u_0 + \frac{\gamma}{2\pi l} \sum_0^\infty \left( \frac{1}{z - \zeta_k} - \frac{1}{z - \zeta'_k} \right) + \frac{\Gamma - \frac{\gamma}{2}}{2\pi l} \frac{1}{z} + f(z),$$

where

$$f(z) = \frac{a_2}{z^2} + \frac{a_3}{z^3} + \dots \quad (2)$$

Consequently,

$$\begin{aligned} \left( \frac{dW}{dz} \right)^2 &= u_0^2 - 2u_0 \frac{\Gamma - \frac{\gamma}{2}}{2\pi l} \frac{1}{z} - \frac{\left( \Gamma - \frac{\gamma}{2} \right)^2}{4\pi^2 l^2} \frac{1}{z^2} - \\ &\quad - 2u_0 \frac{\gamma}{2\pi l} \sum_0^\infty \left( \frac{1}{z - \zeta_k} - \frac{1}{z - \zeta'_k} \right) - \\ &\quad - \frac{\left( \Gamma - \frac{\gamma}{2} \right) \gamma}{2\pi^2 l^2} \frac{1}{z} \sum_0^\infty \left( \frac{1}{z - \zeta_k} - \frac{1}{z - \zeta'_k} \right) - \\ &\quad - \frac{\gamma^2}{4\pi^2 l^2} \left[ \sum_0^\infty \left( \frac{1}{z - \zeta_k} - \frac{1}{z - \zeta'_k} \right) \right]^2 - 2u_0 f(z) + \\ &\quad + 2 \frac{\gamma}{2\pi l} \sum_0^\infty \left( \frac{1}{z - \zeta_k} - \frac{1}{z - \zeta'_k} \right) f(z) + \frac{\Gamma - \frac{\gamma}{2}}{\pi l} \frac{1}{z} f(z) + |f(z)|^2. \end{aligned} \quad (3)$$

Since, on the strength of equality (2),

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$$\int_L f(z) dz = 0, \quad \int_L \frac{1}{z} f(z) dz = 0, \quad \int_L |f(z)|^2 dz = 0$$

and, also,

$$\int_L \sum_{\alpha} \left( \frac{1}{z - \zeta_k} - \frac{1}{z - \zeta'_k} \right) dz = 0,$$

since, within  $L$ , according to the selection of profiles  $L_0$  and  $L$ , there is the same number of points  $\zeta_k$  and  $\zeta'_k$ , for the calculation of

$\int_L \left( \frac{dW}{dz} \right)^2 dz$ , it remains to calculate the integrals

$$\int_L \frac{1}{z} \sum_{\alpha} \left( \frac{1}{z - \zeta_k} - \frac{1}{z - \zeta'_k} \right) dz, \quad (4)$$

$$\int_L f(z) \sum_{\alpha} \left( \frac{1}{z - \zeta_k} - \frac{1}{z - \zeta'_k} \right) dz \quad (5)$$

and, besides, the integral

$$\int_L \left[ \sum_{\alpha} \left( \frac{1}{z - \zeta_k} - \frac{1}{z - \zeta'_k} \right) \right]^2 dz. \quad (6)$$

For calculation of integrals (4) and (5), beforehand, we calculate the integral

$$\int_L f(z) \frac{1}{z - \zeta} dz. \quad (7)$$

We consider two cases:

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1. when point  $\zeta$  is inside  $L$ ;
2. when  $\zeta$  is outside  $L$ .

In the first case, integral (7) equals zero, since profile  $L$  can replace circumference  $C$  of as large a radius  $r$  as desired and, since, at large  $r$ ,  $m$  can be selected, so that

$$|f(z)| < \frac{m}{r^2},$$

then

$$\left| \int_L f(z) \frac{dz}{z - \zeta} \right| < \frac{m}{r^2} \frac{2\pi r}{r - |\zeta|}, \quad (8)$$

and, with unlimited increase of  $r$ , the right side of inequality (8) tends towards zero. If  $\zeta$  is outside  $L$ , after encircling  $\zeta$  with circumference  $c$  of small radius  $h$  and constructing circumference  $C$ ,

which includes  $L$  and  $c$  within it, according to the preceding, we obtain

$$\int_L f(z) \frac{dz}{z-\zeta} = \int_L f(z) \frac{dz}{z-\zeta} + \int_c f(z) \frac{dz}{z-\zeta} = 0,$$

and, since

$$\int_c f(z) \frac{dz}{z-\zeta} = 2\pi i f(\zeta),$$

then

$$\int_L f(z) \frac{dz}{z-\zeta} = -2\pi i f(\zeta). \quad (8')$$

By using equality (8'), we obtain

$$\int_L f(z) \sum_0^{\infty} \left( \frac{1}{z-\zeta_k} - \frac{1}{z-\zeta'_k} \right) dz = 2\pi i \sum_{k=m+1}^{\infty} [f(\zeta'_k) - f(\zeta_k)], \quad (9)$$

in which the summation in the right side of equality (9) extends only to points  $\zeta_k$  and  $\zeta'_k$ , which lie outside  $L$  and, therefore,

$$|\zeta_k| > H^{1/2} + (k-m)l, \quad |\zeta'_k| > H^{1/2} + (k-m)l.$$

Consequently,

$$\left| \sum_{k=m+1}^{\infty} [f(\zeta'_k) - f(\zeta_k)] \right| < 2 \sum_0^{\infty} \frac{1}{(H^{1/2} + kl)^2} < \frac{2}{H^3} + \\ + 2 \int_0^{\infty} \frac{dx}{(H^{1/2} + xl)^2} = \frac{2}{H^3} + \frac{1}{lH^{1/2}}.$$

Thus,

$$\int_L f(z) \sum_{k=0}^{\infty} \left( \frac{1}{z-\zeta_k} - \frac{1}{z-\zeta'_k} \right) dz = O\left(\frac{1}{H^{1/2}}\right). \quad (10)$$

The preceding transformations can be applied to integral (4), and we obtain

$$\int_L \frac{1}{z} \sum_0^{\infty} \left( \frac{1}{z-\zeta_k} - \frac{1}{z-\zeta'_k} \right) dz = 2\pi i \sum_{k=m+1}^{\infty} \left( \frac{1}{\zeta'_k} - \frac{1}{\zeta_k} \right),$$

but

$$\left| \sum_{m+1}^{\infty} \left( \frac{1}{\zeta'_k} - \frac{1}{\zeta_k} \right) \right| = |\zeta'_0 - \zeta_0| \sum_{m+1}^{\infty} \frac{1}{|\zeta_k| |\zeta'_k|} < \\ < |\zeta'_0 - \zeta_0| \sum_0^{\infty} \frac{1}{(H^{1/2} + lk)^2} < |\zeta'_0 - \zeta_0| \left( \frac{1}{H^3} + \frac{1}{lH^{1/2}} \right).$$

With all the results obtained taken into account, after the evident transformations, we have

$$\frac{1}{2} \int_L \left( \frac{dW}{dz} \right)^2 dz = -\pi \left( \Gamma - \frac{1}{2} \right) - \frac{\pi^2}{8} \int_L \left[ \sum_{k=1}^{\infty} \left( \frac{1}{z - z_k} - \frac{1}{z - z'_k} \right) \right]^2 dz + O\left(\frac{1}{H^3}\right). \quad (11)$$

Thus, it remains to calculate the integral

$$\int_L \left[ \sum_{k=1}^{\infty} \left( \frac{1}{z - z_k} - \frac{1}{z - z'_k} \right) \right]^2 dz. \quad (12)$$

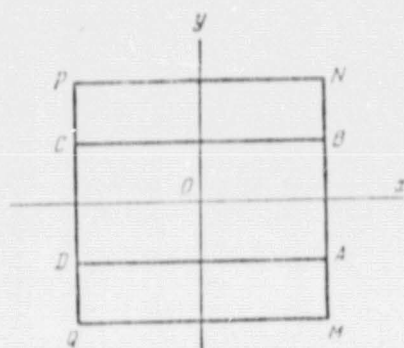


Fig. 17.

We show that integral (12), along line L, can be replaced by the integral with respect to side AB. For this, besides rectangle ABCD with sides AB=2H and BC=2H<sup>3/2</sup>, we construct square MNPQ (Fig. 17) with sides 2H<sup>3/2</sup>. Then, integration along L can be replaced by integration along square MNPQ. Yet, according to inequality (14), Section 6, on sides PN and QM, where |y|=H<sup>3/2</sup>,

$$\left| \frac{\gamma}{2\pi i} \sum_{k=1}^{\infty} \left( \frac{1}{z - z_k} - \frac{1}{z - z'_k} \right) \right| < \frac{\gamma}{2\pi} |z_0 - z'_0| \times \left( \frac{H^{3/2} - r + 1}{H^{3/2} - \frac{h}{2}} + \frac{\pi}{2i} \frac{1}{H^{3/2} - \frac{h}{2}} \right), \quad (13)$$

i.e., such a space M can be selected, that

$$\left| \frac{\gamma}{2\pi i} \sum_{k=1}^{\infty} \left( \frac{1}{z - z_k} - \frac{1}{z - z'_k} \right) \right| < \frac{M}{H^{3/2}},$$

and the same inequality, on the strength of inequality (10), Section 6, is valid on PQ. Consequently,

$$\left| \int_{NPQM} \frac{\gamma}{4\pi i} \left[ \sum_{k=1}^{\infty} \left( \frac{1}{z - z_k} - \frac{1}{z - z'_k} \right) \right]^2 dz \right| < \frac{3M^2}{H^3} 2H^{3/2} = \frac{6M^2}{H^{3/2}}. \quad (14)$$

Further, inequality (13) is valid on BN and on AM and, therefore, /531



$$\left| \int_{\mu\gamma} \frac{\gamma}{4\pi^2} \left[ \sum_{k=1}^n \left( \frac{1}{z-z_k} - \frac{1}{z-z'_k} \right) \right] dz \right| \leq \frac{17}{H^2} |H^2 - H| < \frac{M}{H^2}, \quad (15)$$

and the same inequality is valid for the integral along AM. Thus, we finally have

$$\begin{aligned} & \frac{\gamma^2}{4\pi^2} \int_L \left[ \sum_{k=1}^n \left( \frac{1}{z-z_k} - \frac{1}{z-z'_k} \right) \right]^2 dz = \\ & = \frac{\gamma^2}{4\pi^2} \int_{AB} \left[ \sum_{k=1}^n \left( \frac{1}{z-z_k} - \frac{1}{z-z'_k} \right) \right] dz = O\left(\frac{1}{H^2}\right). \end{aligned} \quad (16)$$

We show further that, in the integrals of equalities (16), function  $G(z)$  can be replaced by  $G_0(z)$ . In fact, according to inequalities (18) and (17), Section 6,

$$\begin{aligned} & \left| \frac{\gamma^2}{4\pi^2} \int_{AB} \left[ \sum_{k=1}^n \left( \frac{1}{z-z_k} - \frac{1}{z-z'_k} \right) \right]^2 dz - \right. \\ & \left. - \frac{\gamma^2}{4\pi^2} \int_{AB} \left[ \sum_{k=1}^n \left( \frac{1}{z-z_k} - \frac{1}{z-z'_k} \right) \right]^2 dz \right| = \\ & = \left| \int_{AB} [G_0(z)]^2 - [G(z)]^2 dz \right| < \\ & < \int_{AB} \frac{\gamma^2}{4\pi^2} \left| z_0 - z'_0 \right| \left[ \frac{1}{H^2 - r} + \frac{1}{l(|y| - \frac{h}{2})} \operatorname{arctg} \frac{|y| - \frac{h}{2}}{H^2 - r} \right] \frac{182}{l^2} dy < \\ & < M \int_{AB} \frac{1}{|y| - \frac{h}{2}} \frac{|y| - \frac{h}{2}}{H^2 - r} dy < M \frac{2H}{H^2 - r} = O\left(\frac{1}{H^2}\right). \end{aligned}$$

Thus, finally, we have, by substituting the values found in equality (11),

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$$\begin{aligned} & \frac{\rho}{2} \int_L \left( \frac{dW}{dz} \right)^2 dz = -\gamma u_0 \left( 1 - \frac{\gamma}{2} \right) - \\ & - \frac{\gamma^2}{8\pi^2} \int_{AB} \left[ \sum_{k=1}^n \left( \frac{1}{z-z_k} - \frac{1}{z-z'_k} \right) \right]^2 dz + O\left(\frac{1}{H^2}\right), \end{aligned} \quad (17)$$

and, since

$$\begin{aligned} & \frac{1}{\pi^2} \left[ \sum_{k=1}^n \left( \frac{1}{z-z_k} - \frac{1}{z-z'_k} \right) \right]^2 = \frac{1}{l^2} \left[ \operatorname{ctg} (z-z_0) \frac{\pi}{l} - \operatorname{ctg} (z-z'_0) \frac{\pi}{l} \right]^2 = \\ & = \frac{1}{l^2} \frac{\sin^2(z_0 - z'_0) \frac{\pi}{l}}{\sin^2(z-z_0) \frac{\pi}{l} \sin^2(z-z'_0) \frac{\pi}{l}}. \end{aligned}$$

besides,

$$\begin{aligned} \frac{p}{2} \int_L \left( \frac{dW}{dz} \right)^2 dz = \\ = -\gamma u_0 \left( 1 - \frac{\gamma}{2} \right) - \frac{\gamma^2}{8l^2} \int \frac{\sin^2(z_0 - z'_0) \frac{\pi}{l}}{\sin^2(z - z_0) \frac{\pi}{l} \sin^2(z - z'_0) \frac{\pi}{l}} dz + O\left(\frac{1}{H}\right). \end{aligned} \quad (18)$$

Finally, we show that the integral with respect to AB in equality (18) can be replaced by the integral over the entire infinite straight line  $x=H^{3/2}$ .

Actually, by repeating the transformation of Section 8, we show that

$$\begin{aligned} \left| \sin^2(z - z_0) \frac{\pi}{l} \sin^2(z - z'_0) \frac{\pi}{l} \right| = \\ = \frac{1}{4} \left[ \cos 2i(y - \eta) \frac{\pi}{l} - \cos 2(x - \xi) \frac{\pi}{l} \right] \times \\ \times \left[ \cos 2i(y - \eta') \frac{\pi}{l} - \cos 2(x - \xi') \frac{\pi}{l} \right] = \\ = \frac{1}{16} \left[ e^{2i(y - \eta) \frac{\pi}{l}} + e^{-2i(y - \eta) \frac{\pi}{l}} - 2 \cos 2(x - \xi) \frac{\pi}{l} \right] \times \\ \times \left[ e^{2i(y - \eta') \frac{\pi}{l}} + e^{-2i(y - \eta') \frac{\pi}{l}} - 2 \cos 2(x - \xi') \frac{\pi}{l} \right] = \\ = \frac{1}{16} e^{4i\eta l} \frac{\pi}{l} \left[ e^{-2\xi \frac{\pi}{l}} + e^{-2\xi' \frac{\pi}{l}} \right] + O(e^{l\eta l}) = \\ = \frac{1}{8} e^{4i\eta l} \frac{\pi}{l} \operatorname{ch} \frac{\eta \pi}{2l} + O(e^{l\eta l}), \end{aligned}$$

and, therefore, the integral

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$$\left| \int_H^{i\infty} \frac{\sin^2(z_0 - z'_0) \frac{\pi}{l}}{\sin^2(z - z_0) \frac{\pi}{l} \sin^2(z - z'_0) \frac{\pi}{l}} dz \right| < \int_H^\infty M e^{-ls} ds = \frac{M}{4} e^{-lH}.$$

Thus, we can rewrite inequality (18) in the form

$$\begin{aligned} \frac{p}{2} \int_L \left( \frac{dW}{dz} \right)^2 dz = -\gamma u_0 \left( 1 - \frac{\gamma}{2} \right) - \\ - \frac{\gamma^2}{8l^2} \int_{-\infty}^{\infty} \frac{\sin^2(z_0 - z'_0) \frac{\pi}{l}}{\sin^2(z - z_0) \frac{\pi}{l} \sin^2(z - z'_0) \frac{\pi}{l}} dz + O\left(\frac{1}{H^{3/2}}\right). \end{aligned} \quad (19)$$

The method of calculation of the last integral is well known. For example, it can be calculated in the following manner. Initially, we calculate the integral

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$$\begin{aligned}
& \int_{-1}^1 \frac{\sin(\zeta_0 - \zeta'_0) \frac{\pi}{l}}{\sin(z - \zeta_0) \frac{\pi}{l} \sin(z - \zeta'_0) \frac{\pi}{l}} dz = \\
& = \int_{-1}^1 \left[ \operatorname{ctg} \frac{\pi}{l} (z - \zeta_0) - \operatorname{ctg} \frac{\pi}{l} (z - \zeta'_0) \right] dz = \\
& = \frac{l}{\pi} \ln \frac{\sin(\zeta_0 - \zeta'_0) \frac{\pi}{l}}{\sin(\zeta_0 - \zeta'_0) \frac{\pi}{l}} \Big|_{-1}^1 = \\
& = \frac{l}{\pi} \ln \frac{e^{\frac{\pi}{l}(\zeta_0 - \zeta'_0)} - e^{-\frac{\pi}{l}(\zeta_0 - \zeta'_0)}}{e^{\frac{\pi}{l}(\zeta_0 - \zeta'_0)} - e^{-\frac{\pi}{l}(\zeta_0 - \zeta'_0)}} \Big|_{-1}^1 = \\
& = \frac{l}{\pi} \ln e^{\frac{\pi}{l}(\zeta_0 - \zeta'_0)} - \frac{l}{\pi} \ln e^{-\frac{\pi}{l}(\zeta_0 - \zeta'_0)} = 2(\zeta_0 - \zeta'_0)l.
\end{aligned}$$

In this manner,

$$\int_{-1}^1 \frac{dz}{\sin(z - \zeta_0) \frac{\pi}{l} \sin(z - \zeta'_0) \frac{\pi}{l}} = 2l \frac{\zeta_0 - \zeta'_0}{\sin(\zeta_0 - \zeta'_0) \frac{\pi}{l}}. \quad (20)$$

By differentiating equality (20) with respect to parameters  $\zeta_0$  and  $\zeta'_0$ , we obtain

$$\begin{aligned}
& \int_{-1}^1 \frac{\cos(z - \zeta_0) \frac{\pi}{l} \sin(z - \zeta'_0) \frac{\pi}{l}}{\sin^2(z - \zeta_0) \frac{\pi}{l} \sin^2(z - \zeta'_0) \frac{\pi}{l}} dz = \\
& = 2l \frac{l}{\pi} \frac{\sin(\zeta_0 - \zeta'_0) \frac{\pi}{l} - \frac{\pi}{l} (\zeta_0 - \zeta'_0) \cos(\zeta_0 - \zeta'_0) \frac{\pi}{l}}{\sin^2(\zeta_0 - \zeta'_0) \frac{\pi}{l}}
\end{aligned}$$

and

$$\begin{aligned}
& \int_{-1}^1 \frac{\cos(z - \zeta'_0) \frac{\pi}{l} \sin(z - \zeta_0) \frac{\pi}{l}}{\sin^2(z - \zeta_0) \frac{\pi}{l} \sin^2(z - \zeta'_0) \frac{\pi}{l}} dz = \\
& = -2l \frac{l}{\pi} \frac{\sin(\zeta_0 - \zeta'_0) \frac{\pi}{l} - \frac{\pi}{l} (\zeta_0 - \zeta'_0) \cos(\zeta_0 - \zeta'_0) \frac{\pi}{l}}{\sin^2(\zeta_0 - \zeta'_0) \frac{\pi}{l}},
\end{aligned}$$

and, by calculating the second equality from the first and multiplying by  $\sin(\zeta_0 - \zeta'_0) \frac{\pi}{l}$ , we have

$$\begin{aligned}
& \int_{-1}^1 \frac{\sin^2(\zeta_0 - \zeta'_0) \frac{\pi}{l}}{\sin^2(z - \zeta_0) \frac{\pi}{l} \sin^2(z - \zeta'_0) \frac{\pi}{l}} dz = \\
& = 4l \frac{l}{\pi} \left[ 1 - (\zeta_0 - \zeta'_0) \frac{\pi}{l} \operatorname{ctg}(\zeta_0 - \zeta'_0) \frac{\pi}{l} \right].
\end{aligned}$$

At the selected location of line  $L_0$ , we have

and, therefore,

$$\begin{aligned} (z_0 - z_0') \frac{\pi}{l} \sin^2(z_0 - z_0') \frac{\pi}{l} &= \left( -\frac{\pi}{2} + \frac{h\pi}{l} i \right) \operatorname{th} \left( -\frac{\pi}{2} + \frac{h\pi}{l} i \right) \\ &= -\left( -\frac{\pi}{2} + \frac{h\pi}{l} i \right) i \operatorname{th} \frac{h\pi}{l}. \end{aligned}$$

Consequently,

$$\begin{aligned} \int_{-l}^l \frac{\sin^2(z_0 - z_0') \frac{\pi}{l}}{\sin^2(z - z_0) \frac{\pi}{l} \sin^2(z - z_0') \frac{\pi}{l}} dz &= \\ &= 4l \left[ 1 + \left( -\frac{\pi}{2} i + \frac{h\pi}{l} \right) \operatorname{th} \frac{h\pi}{l} \right], \end{aligned}$$

and, by substituting in Eq. (19), we have

$$\begin{aligned} \frac{\rho}{2} \int_{-l}^l \left( \frac{dW}{dz} \right)^2 dz &= -\rho u_0 \left( \Gamma - \frac{\pi}{2} \right) - \frac{\rho V_1^2}{2\pi l} \left[ \frac{\pi}{2} \operatorname{th} \frac{h\pi}{l} + \right. \\ &\quad \left. + i \left( 1 - \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} \right) \right] + o\left(\frac{1}{H^2}\right), \end{aligned}$$

or

$$\begin{aligned} \frac{\rho}{2} \int_{-l}^l \left( \frac{dW}{dz} \right)^2 dz &= -\rho u_0 \Gamma + \rho u_0 \frac{\pi}{2} - \frac{\rho V_1^2}{4l} \operatorname{th} \frac{h\pi}{l} + \\ &\quad + i \frac{\rho V_1^2}{2\pi l} \left[ \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1 \right] + o\left(\frac{1}{H^2}\right). \end{aligned}$$

Yet, according to formula (8), Section 4,

$$u_0 = \frac{\pi}{2l} \operatorname{th} \frac{h\pi}{l},$$

and, therefore,

$$\frac{\pi}{2} \frac{V_1^2}{4l} \operatorname{th} \frac{h\pi}{l} = \rho u_0 \frac{\pi}{2},$$

and, consequently, we finally have

$$\frac{\rho}{2} \int_{-l}^l \left( \frac{dW}{dz} \right)^2 dz = -\rho u_0 \Gamma + i \frac{\rho V_1^2}{2\pi l} \left[ \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1 \right] + o\left(\frac{1}{H^2}\right). \quad (21)$$

and, from this, since the first terms of the second part of equality (21) do not depend on time,



$$\frac{\rho}{2} \int_0^T dt \int_L \left( \frac{dW}{dz} \right)^2 dz = -\rho a_0 VT + \frac{\rho \gamma^2}{2\pi l} \left[ \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1 \right] T + O\left(\frac{1}{H^2}\right). \quad (22)$$

### Section 10. Lift and Thrust of Flapping Wing

We now turn to basic formula (1), Section 4,

$$(Y_0 + iX_0)T = -\frac{\rho}{2} \int_0^T dt \int_L \left( \frac{dW}{dz} \right)^2 dz - \rho \int_L |\gamma|_0^2 (dx - i dy) = \rho l \left[ \int_S \int \frac{dW}{dz} dx dy \right]_0^T. \quad (1)$$

Since we have found

$$\begin{aligned} 1. \quad & \frac{\rho}{2} \int_0^T dt \int_L \left( \frac{dW}{dz} \right)^2 dz = -\rho a_0 VT + \frac{\rho \gamma^2}{2\pi l} \left( \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1 \right) T + O\left(\frac{1}{H^2}\right) \quad ((22), \text{Sec. } 9), \\ 2. \quad & \int_L |\gamma|_0^2 (dx - i dy) = \Pi + O\left(\frac{1}{H^2}\right) \quad ((4), \text{Sec. } 7), \\ 3. \quad & \left[ \int_S \int \frac{dW}{dz} dx dy \right]_0^T = \gamma h + O\left(\frac{1}{H^2}\right) \quad ((10), \text{Sec. } 8), \end{aligned}$$

by substituting the values found in (1) and upon noting that  $\mathfrak{L} = (V + u_0)T$ , after division by  $T$ , we obtain

$$Y_0 + iX_0 = -\rho a_0 V - i \frac{\rho \gamma^2}{2\pi l} \left( \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1 \right) - \rho l (V + u_0) - \frac{\rho \gamma h}{l} + O\left(\frac{1}{H^2}\right) \quad (2)$$

and, proceeding to the limit, by increasing the dimensions without limit, we obtain the final equation for determination of the average value of the forces on the flapping wing, /537

$$Y_0 + iX_0 = -\rho VT - i \left[ \frac{\rho \gamma^2}{2\pi l} \left( \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1 \right) + \frac{\rho \gamma h}{l} (V + u_0) \right]. \quad (3)$$

From which

$$X_0 = -\frac{\rho \gamma^2}{2\pi l} \left( \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1 \right) - \frac{\rho \gamma h}{l} (V + u_0), \quad (4)$$

$$Y_0 = -\rho VT. \quad (5)$$

With respect to the corrected derivation of Eq. (3), we note that the idea of it is close to the method of Syngé (1) which, for the case of a cylinder with a double vortex street shed from it, Eq. (3) was given for the first time. However, the derivation of Syngé contains an obvious error.<sup>1</sup> Moreover, Syngé, for the potential of a vortex street which is infinite in one direction, uses the properties of function  $\Gamma(x)$ . The corrected derivation shows that this is completely superfluous. As was shown in Section 6, it is sufficient to select profile L in a suitable manner.

Eq. (5) evidently gives an analog of the Zhukovskiy theorem in the case under consideration. Since

$$\Gamma = \frac{1}{r} (\Gamma_1 \frac{r}{2} + \Gamma_2 \frac{r}{2}) = \frac{1}{2} (\Gamma_1 + \Gamma_2),$$

by noting that, according to formulas (3) and (4), Section 4,

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$$\Gamma_1 = -\pi b \sqrt{V^2 + w^2} \sin(\alpha_1 + \arctg \frac{w}{V}),$$

$$\Gamma_2 = -\pi b \sqrt{V^2 + w^2} \sin(\alpha_2 - \arctg \frac{w}{V}),$$

we find that

$$\Gamma = -\pi b \sqrt{V^2 + w^2} \sin \tau \cos(\delta + \arctg \frac{w}{V}),$$

where

$$\tau = \frac{\alpha_1 + \alpha_2}{2}, \quad \delta = \frac{\alpha_1 - \alpha_2}{2} \quad \text{or} \quad \delta = \frac{\alpha}{2} + \theta.$$

From this, we finally obtain<sup>2</sup>

$$\Gamma = -\pi b V \sin \tau [\cos \delta - \sin \delta \frac{w}{V}]. \quad (6)$$

In particular, if  $\theta_1 = \theta_2 = \theta$  and, consequently,  $\mu_1 = \mu_2 = \theta + \frac{\alpha}{2}$ ,

$$\Gamma = -\pi b V \sin(\frac{\alpha}{2} + \theta). \quad (7)$$

1

In the derivation of Syngé, L is fixed. Here, because of periodicity of the flow,

$$\left(\frac{\partial \tilde{\varphi}}{\partial x}\right)_{t=0} = \left(\frac{\partial \tilde{\varphi}}{\partial x}\right)_{t=T}, \quad \left(\frac{\partial \tilde{\varphi}}{\partial y}\right)_{t=0} = \left(\frac{\partial \tilde{\varphi}}{\partial y}\right)_{t=T},$$

from which it follows that  $[\tilde{\varphi}]_0^T = \text{const}$  and  $\int_L [\tilde{\varphi}]_0^T (dx - l dy) = 0$ , while, in Syngé, we have

$$\int_L [\tilde{\varphi}]_0^T (dx - l dy) = \gamma h.$$

2

We note that, as calculations show, in formula (6),  $\cos \delta - \sin \delta \frac{w}{V}$  can be greater than one, so that the coefficient of lift of a flapping wing is somewhat higher than that of a stationary wing.

i.e.,  $\Gamma$  is the circulation which the wing has in the absence of flapping.

We transform equality (4). Since

$$u_0 = \frac{\gamma}{2l} \operatorname{th} \frac{h\pi}{l},$$

then

$$X_0 = \frac{\rho V^2}{2\pi l} - \frac{\rho \gamma h}{l} (V + 2u_0). \quad (8)$$

Formula (8) completely coincides with the formula given by Karman for calculation of the frontal drag, if the signs of  $\gamma$  and  $u_0$  are changed in it. This had to be expected since, from the point of view in which the present work was built up, the difference of thrust from the drag due to the vortices of the street is only a change in rotation of those vortices which form the street, and the change of sign of  $u_0$  connected with this. /539

If the vortex street stability conditions

$$\frac{h}{l} = 0.281 \quad \text{and} \quad \gamma = lV\sqrt{8}u_0$$

are substituted in Eq. (8), we obtain the thrust in the form

$$X_0 = -\rho V^2 \left[ 0.314 \left( \frac{u_0}{V} \right)^2 + 0.794 \frac{u_0}{V} \right]; \quad (9)$$

Eq. (9) corresponds to the known Karman formula for drag.

In the case of the problem under consideration, the magnitude of the thrust must be expressed through the data which defines the shape and motion of the wing. Since, according to formula (6), Section 4,

$$\gamma = 2\pi b \cos \alpha [\sin \alpha V + \cos \alpha \omega] \quad (10)$$

and, according to formula (9), Section 4,

$$u_0 = \frac{b}{h} \cos \alpha [\sin \alpha V + \cos \alpha \omega] \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l}, \quad (11)$$

by substituting these values in formula (4), we obtain

$$X_0 = -2\pi \rho b \cos \alpha (V \sin \alpha + \cos \alpha \omega) \times \\ \times \left[ \frac{b}{l} \cos \alpha (\sin \alpha V + \cos \alpha \omega) \left\{ 2 \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1 \right\} + \frac{h}{l} V \right],$$

or

$$X_0 = -2\pi \rho b V^2 \cos \alpha \left( \sin \alpha + \cos \alpha \frac{\omega}{V} \right) \frac{h}{l} \times \\ \times \left[ 1 + \frac{b}{h} \cos \alpha \left( \sin \alpha + \cos \alpha \frac{\omega}{V} \right) + \left\{ 2 \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1 \right\} \right]. \quad (12)$$

In particular, if  $\theta_1 = \theta_2 = 0$ , when  $\sigma = \frac{\alpha}{2} + a$  and  $\delta = 0$ ,

$$X_0 = -2\pi\rho bVw \cos\left(\frac{\pi}{2} + \delta\right) \frac{h}{l} \times \\ \times \left[1 + \frac{b}{h} \cos\left(\frac{\pi}{2} + \delta\right) \frac{w}{V} \left\{2 \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1\right\}\right]. \quad (13)$$

In satisfaction of the stability conditions of the street, from 7540 (12), we have

$$X_0 = -1,76\rho bV^2 \cos z \left(\sin \delta + \cos \delta \frac{w}{V}\right) \times \\ \times \left[1 + 0,25 \frac{b}{h} \cos z \left(\sin \delta + \cos \delta \frac{w}{V}\right)\right]. \quad (14)$$

Since, in satisfaction of the stability condition, from Eq. (12), Section 4,

$$\frac{b}{h} = \frac{\frac{w}{V} - 0,562}{0,35 \cos z \left[\sin \delta + \cos \delta \frac{w}{V}\right]},$$

finally, from (14), we have<sup>3</sup>

$$X_0 = -0,94\rho bV^2 \cos z \left(\sin \delta + \cos \delta \frac{w}{V}\right) \left[1 + 1,2 \frac{w}{V}\right]. \quad (15)$$

To return to the general case, we note that, from Eq. (10), Section 4,

$$\frac{b}{h} = \frac{\frac{w}{V} - 2 \frac{h}{l}}{2 \frac{h}{l} \left[\frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l}\right] \cos z \left[\sin \delta + \cos \delta \frac{w}{V}\right]}, \quad (16)$$

and, by substituting this value in expression (12), we obtain

$$X_0 = -\pi\rho bV^2 \cos z \left(\sin \delta + \cos \delta \frac{w}{V}\right) \times \\ \times \frac{2 \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1}{\frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l}} \left\{ \frac{w}{V} - 2 \frac{h}{l} \frac{\frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1}{2 \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1} \right\}. \quad (17)$$

---

<sup>2</sup> We note that, in Eq. (15),  $\frac{w}{V} > 0,562$ .



The expression for thrust can be given another form. First, while noting that the circulation of the vortices which form the street is expressed by the formula

$$\gamma = 2\pi b \cos z (\sin 2W + \cos 2w),$$

we can rewrite equality (17) in the form

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$$X_0 = -\rho V l \left[ \frac{2 \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1}{2 \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l}} \left[ \frac{w}{V} - 2 \frac{h}{l} \right] - 2 \frac{h}{l} \frac{\frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1}{2 \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l}} \right] \quad (17')$$

Finally, from Eq. (16), we can obtain

$$b \cos z \left[ \sin 2z + \cos 2z \frac{w}{V} \right] = h \left[ \frac{w}{V} - 2 \frac{h}{l} \right] \frac{1}{2 \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l}},$$

and, therefore, equality (17) can be written further in the form

$$X_0 = -\pi \rho V^2 l \left[ \frac{w}{V} - 2 \frac{h}{l} \right] \times \\ \times \left[ \frac{w}{V} - 2 \frac{h}{l} \frac{\frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1}{2 \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l}} \right] \frac{2 \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1}{2 \left( \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} \right)^2} \quad (17'')$$

In all these expressions, the necessity of satisfaction of relationship (16) should be remembered. The latter relationship, by setting

$$\frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} = M,$$

can be rewritten in the form

$$X_0 = -\pi \rho V^2 l \left[ \frac{w}{V} - 2 \frac{h}{l} \right] \left[ \frac{w}{V} - 2 \frac{h}{l} \frac{M-1}{2M-1} \right] \frac{2M-1}{M} \frac{1}{2M},$$

and, since

$$\frac{M-1}{2M-1} = 1 - \frac{M}{2M-1},$$

we have, further,

$$X_0 = -\pi \rho V^2 l \left[ \frac{w}{V} - 2 \frac{h}{l} \right] \left[ \left( \frac{w}{V} - 2 \frac{h}{l} \right) \frac{2M-1}{M} + 2 \frac{h}{l} \right] \frac{1}{2M} \quad (17''')$$

Since the data which define the work of a flapping wing are the quantities  $b$ ,  $\mu_1 = \frac{\alpha}{2} + \theta_1$  and  $\mu_2 = \frac{\alpha}{2} + \theta_2$ , from which  $\sigma = \frac{\mu_1 + \mu_2}{2}$  and  $\delta = \frac{\mu_1 - \mu_2}{2}$ ,

as well as quantities  $b/h$  and  $w/V$  are determined, for calculation of the thrust from Eq. (16),  $h/l$  must be found and, from Eq. (17),  $X_0$ .

Here, we have to remember that, for the possibility of the existence of flow around the wing, from the physical point of view, the following condition must be satisfied

$$\frac{w}{V} > 2 \frac{h}{l}. \quad (18)$$

We consider the expression

$$\begin{aligned} f(M) &= \frac{2M-1}{M} \left[ \frac{w}{V} - 2 \frac{h}{l} \frac{M-1}{2M-1} \right] = \\ &= \frac{(2M-1) \frac{w}{V} - 2 \frac{h}{l} (M-1)}{M}. \end{aligned} \quad (19)$$

Since

$$(2M-1) \frac{w}{V} - 2 \frac{h}{l} (M-1) = (2M-1) \left( \frac{w}{V} - 2 \frac{h}{l} \right) + 2 \frac{h}{l} M,$$

from condition (18), it follows that, with  $2M-1 > 0$ ,  $f(M) > 0$  and, consequently,  $X_0 < 0$ , i.e., with  $M = \frac{h/l}{2} > \frac{1}{2}$ , we have the thrust. This condition is satisfied at  $\frac{h}{l} > 0.245$ . Thus, with  $\frac{h}{l} > 0.245$ , we have the thrust with any  $w/V$  ratio which satisfies condition (18). From Eq. (19), we see that  $X_0 = 0$  with

$$\frac{w}{V} = 2 \frac{h}{l} \frac{M-1}{2M-1} = 2 \frac{h}{l} \frac{1-M}{1-2M}.$$

Since condition (18) occurs, this equation has a solution which satisfies condition (18), only when  $2M < 1$ . In this case,

$$\frac{w}{V} = 2 \frac{h}{l} \left( 1 + \frac{M}{1-2M} \right). \quad (20)$$

Since

$$f(M) = \frac{2M-1}{M} \left[ \frac{w}{V} - 2 \frac{h}{l} \left( 1 + \frac{M}{1-2M} \right) \right],$$

with

$$2 \frac{h}{l} < \frac{w}{V} < 2 \frac{h}{l} \left( 1 + \frac{M}{1-2M} \right) \quad (21)$$

$f(M) < 0$ , and, therefore, from formula (17)

$$X_a > 0.$$

i.e., with  $M < \frac{1}{2}$  or with  $\frac{h}{\lambda} = 0.245$  and with  $w/V$  which satisfies condition (21), the flapping wing does not produce thrust, but drag. We note that such a case only can occur, under conditions which are far from satisfying the vortex street stability conditions. In satisfaction of the stability condition ( $\frac{h}{\lambda} = 0.281$ ), we obtain the thrust with any  $\frac{w}{V} > 0.562$ . In this case, of course, we consider that

$$\sin \delta + \frac{w}{V} \cos \delta > 0.$$

In particular, with  $\delta = 0$ , when  $\theta_1 = \theta_2$ ,

$$X_a = -\pi \rho h w \cos\left(\frac{\pi}{2} + \theta\right) \frac{2M-1}{M} \left[ \frac{w}{V} - 2 \frac{h}{l} \frac{M-1}{2M-1} \right]. \quad (22)$$

## 11. Suction Effect

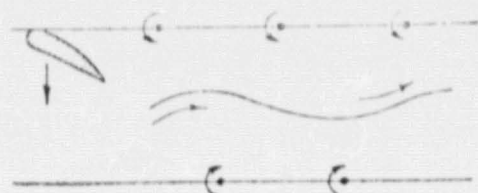


Fig. 18.

The theorem of pulsed force precisely specifies the mechanism of development of the forces which act on a body submerged in a fluid. From this point of view, the generation of the thrust of a flapping wing, as was stated above, has the following simple mechanical meaning. The inverted double vortex street formed behind a flapping wing (Fig. 18) produces additional velocity in the direction of the

windstream. This velocity and the momentum along the flow connected with it are the result of the application of the force applied to the fluid in the direction of the flow velocity and, consequently, equal to it in magnitude, but directed opposite to the windstream velocity, of the force applied to the flapping wing. This is the thrust. In whatever manner we attempt to explain the development of thrust, in any case, according to the theorem of impulses, this force should induce the backward deflection of the mass of fluid. From this point of view, we attempt to explain the essence of the effect of the so called "suction" forces on the wing.

We base all the considerations on the method of accounting for suction forces of N.Ye. Zhukovskiy, in which we restrict ourselves to the case of a flat plate [7]. We imagine a flow past a plate (Fig. 19), such that the trailing edge B is the point the flow leaves the plate. In this case, a physically impossible infinite velocity is produced around the leading edge. The idea of Zhukovskiy is that a vortex of

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such intensity  $J$  is produced very close to the leading edge, that at point A itself, the velocity is finite, and the air flow past the wing flows past the plate and a certain turbulent region around the leading edge. It follows from this that the pressure of the flow past the plate consists of the pressure on the air filled turbulent region, which supplements the wing, and of the pressure on the wing itself. We calculate both pressures.

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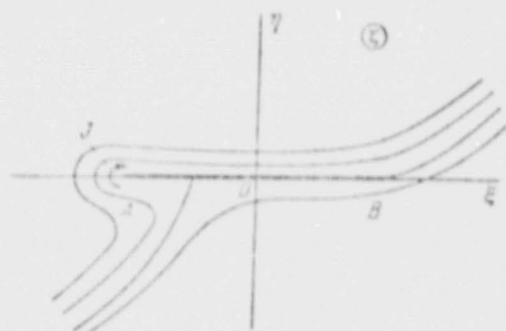


Fig. 19.

We represent, by means of the function

$$\zeta = z + \frac{R^2}{z} \quad (1)$$

the surface portion of circle  $|z|=R$  on the surface of the plate. In this case,

radius  $R$  of the circle will equal  $b/4$ , where  $b$  is the width (chord) of the plate.

The flow past a round cylinder is given by the complex potential

$$W = Ve^{-i\alpha} \left( z + \frac{R^2 e^{2i\alpha}}{z} \right) + \frac{\Gamma}{2\pi i} \ln z. \quad (2)$$

From Eq. (1) and (2), for the complex velocity of the flow past the plate, we find the expression

$$\frac{dW}{dz} = V \left( \cos \theta \pm \sqrt{\frac{b-2z}{b+2z}} \sin \theta \right), \quad (3)$$

in which, for the top surface of the plate, the square root is included with a plus sign and, for the lower surface, with a minus sign. The pressure on the plate from below ( $p_H$ ) and from the top ( $p_B$ ) is

expressed by the formulas

$$p_H = C - \frac{\rho}{2} V_H^2, \\ p_B = C - \frac{\rho}{2} V_B^2,$$

and, from this, the pressure directed upward normal to the surface of the plate,

$$\Delta p = p_H - p_B = \frac{\rho}{2} [V_H^2 - V_B^2] = \frac{\rho}{2} 2V \cos \theta 2V \sin \theta \sqrt{\frac{b-2z}{b+2z}},$$

and, therefore, the entire pressure of the flow on the plate  $P_1$  is directed normal and upward, and it is



$$P_1 = 2\rho V^2 \sin \theta \cos \theta \int_{-b/2}^{+b/2} \sqrt{\frac{b-z}{b+z}} dz = \pi \rho V^2 b \sin \theta \cos \theta. \quad (4)$$

It is easy to verify that force  $P_1$  is applied at point  $-b/4$ . /546

We now calculate force  $P_2$  applied to vortex  $J$ , which forms at a point close to the leading edge of the wing. We examine the characteristic function of flow past a round cylinder, in the form

$$W = Ve^{-i\theta} \left( z + \frac{R^2 e^{2i\theta}}{z} \right) + \frac{\Gamma}{2\pi i} \ln z + \frac{J}{2\pi i} \left[ \ln(z + R(1+\epsilon)) - \ln \left( z + \frac{R}{1+\epsilon} \right) \right], \quad (5)$$

where  $\Gamma$  is the primary circulation around the wing and  $J$  is the circulation of the vortex at point  $-R(1+\epsilon)$  around the leading edge, so that  $\epsilon$  is a small quantity.

From (5), we have

$$\frac{dW}{dz} = Ve^{-i\theta} \left( 1 - \frac{R^2 e^{2i\theta}}{z^2} \right) + \frac{\Gamma}{2\pi i} \frac{1}{z} + \frac{J}{2\pi i} \left[ \frac{1}{z + R(1+\epsilon)} - \frac{1}{z + \frac{R}{1+\epsilon}} \right]. \quad (6)$$

For the velocity around the leading and trailing edges of the wing to be finite at the plate, it is necessary that, with  $z = \pm R \frac{dw}{dz}$  revert to zero. From this, we have the equation

$$\begin{aligned} Ve^{-i\theta} (1 - e^{2i\theta}) + \frac{\Gamma}{2\pi i R} + \frac{J}{2\pi i R} \left( \frac{1}{2+\epsilon} - \frac{1+\epsilon}{2+\epsilon} \right) &= 0, \\ Ve^{-i\theta} (1 - e^{2i\theta}) - \frac{\Gamma}{2\pi i R} + \frac{J}{2\pi i R} \left( \frac{1}{\epsilon} + \frac{1+\epsilon}{\epsilon} \right) &= 0, \end{aligned} \quad (7)$$

from which

$$J = -4\pi V R \sin \theta \frac{2+\epsilon}{2+2\epsilon} \epsilon,$$

or, approximately, according to the smallness of  $\epsilon$

$$J = -4\pi V R \sin \theta. \quad (8)$$

We now calculate the velocity of the flow along the axis of

J. For this, the following expression must be found

$$\frac{d}{dz} \left( W - \frac{J}{2\pi l} \ln(\zeta - p) \right)_{\zeta=p}, \quad (9)$$

where  $\mu$  is the point of plane  $\zeta$  which corresponds to  $-R(1+\epsilon)$ . Let  $\xi = \chi(z)$  be the representation given by formula (1). Then with  $-R(1+\epsilon)$  designated by  $a$ , we have

$$\zeta - p = \chi(z) - \chi(a) = (z - a)\chi'(a) + \frac{(z - a)^2}{2}\chi''(a) + (z - a)^3(\dots)$$

and

$$\ln(\zeta - p) = \ln(z - a) + \ln \chi'(a) + \frac{z - a}{2} \frac{\chi''(a)}{\chi'(a)} + (z - a)^2(\dots).$$

From this, with the components of the velocity on the axes of the vortex designated  $u$  and  $v$ , we have

$$u - iv = \frac{d}{dz} \left[ V e^{-i\theta} \left( z + \frac{R^2 e^{2i\theta}}{z} \right) + \frac{\Gamma}{2\pi l} \ln z - \frac{J}{2\pi l} \ln \chi'(a) - \frac{J}{2\pi l} \frac{z - a}{2} \frac{\chi''(a)}{\chi'(a)} - \frac{J}{2\pi l} (z - a)^2(\dots) - \frac{J}{2\pi l} \ln \left( z + \frac{R}{1 + \epsilon} \right) \right]_{z=a} \frac{dz}{dz}.$$

Otherwise,

$$u - iv = \left[ V e^{-i\theta} \left( 1 - \frac{R^2 e^{2i\theta}}{z^2} \right) + \frac{\Gamma}{2\pi l} \frac{1}{z} - \frac{J}{2\pi l} \frac{1}{2} \frac{\chi''(a)}{\chi'(a)} - \frac{J}{2\pi l} \frac{1}{z + \frac{R}{1 + \epsilon}} \right]_{z=a} \left( \frac{dz}{dz} \right)_{z=a}.$$

Since  $a = -R(1 + \epsilon)$ ,

$$V e^{-i\theta} \left( 1 - \frac{R^2 e^{2i\theta}}{z^2} \right) + \frac{\Gamma}{2\pi l} \frac{1}{z} \Big|_{z=a} = V e^{-i\theta} \left( 1 - \frac{e^{2i\theta}}{(1 + \epsilon)^2} \right) - \frac{\Gamma}{2\pi l R} \frac{1}{1 + \epsilon},$$

which, according to the smallness of  $\epsilon$ , can further be written in the form

$$V e^{-i\theta} (1 - e^{2i\theta}) - \frac{\Gamma}{2\pi l R},$$

and, on the strength of Eq. (7),

$$V e^{-i\theta} (1 - e^{2i\theta}) - \frac{\Gamma}{2\pi l R} = -\frac{J}{2\pi l R} \frac{2 + \epsilon}{\epsilon} \approx -\frac{J}{2\pi l R} \frac{1}{\epsilon}.$$

Further, since  $\chi(z) = z + \frac{R^2}{z}$ , to  $\chi'(z) = \frac{2R^2}{z(z^2 - R^2)}$ , therefore,

$$\frac{\chi''(z)}{\chi'(z)} = \frac{2R^2}{-R^2(1+z)(2z+z^3)} \approx -\frac{1}{R^2}.$$

In exactly the same way,

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$$\left( \frac{1}{z + \frac{R^2}{1+z}} \right)_{z=a} = -\frac{1+z}{R(2z+z^3)} \approx -\frac{1}{2R^2}.$$

Finally,

$$\left( \frac{dz}{\chi'(z)} \right)_{z=a} = \left( \frac{z^3}{z^3 - R^2} \right)_{z=a} = \frac{(1+z)^2}{2z+z^3} \approx \frac{1}{2z}.$$

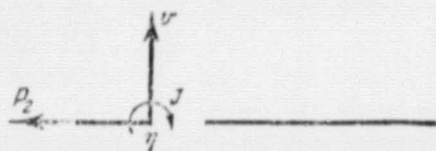
It follows from this that

$$u - iv = \left[ -\frac{J}{2iR} \frac{1}{z} + \frac{J}{2iR} \frac{1}{2z} + \frac{J}{2iR} \frac{1}{2z} \right] \frac{1}{2z} = -\frac{J}{2iRz} \frac{1}{2z} = -\frac{J}{4iRz^2}, \text{ i.e. } v = \frac{J}{4\pi R z^2}.$$

To calculate force  $P_2$ , we now apply the Zhukovskiy theorem to the mass of fluid enveloped by the flow around the leading edge. Then, for force  $P_2$ , we obtain the expression

$$P_2 = \rho v |J| = \frac{J^2}{4\pi R^2} = 4\pi \rho R V^2 \sin^2 \theta. \quad (10)$$

We obtain the direction of this force from the Zhukovskiy theorem, as shown in Fig. 20.



In summarizing everything stated above, force  $P$  acts on the plate. Its value is determined by the formula

$$P = \sqrt{P_1^2 + P_2^2} = 4\pi \rho R V^2 \sin \theta = \pi \rho b V^2 \sin \theta,$$

Fig. 20.

and this force is directed perpendicular to  $V$ , since

$$\operatorname{tg} \alpha = \frac{P_2}{P_1} = \operatorname{tg} \theta \quad \text{and} \quad \alpha = \theta \quad (\text{Fig. 21}).$$

It follows from this that the force determined from the Zhukovskiy theorem (or, which is the same thing, from the Chaplygin-Blasius formula) includes the suction and, therefore, to take account of the suction for a flapping wing, it is sufficient only to take account of the

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effect of the force determined by the Zhukovskiy formula.

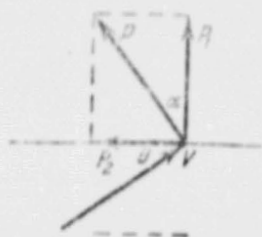


Fig. 21.

Thus, if we construct a theory of flapping wings from the physical scheme of total flow past a wing without current formation, as all conventional theories of oscillating or flapping wings are constructed, i.e., from the forces calculated on the basis of the Zhukovskiy theorem, there is no basis for speaking of suction.

Based on the Zhukovskiy theorem and without taking account of the effect of the departing vortices, the following very rough estimate of the magnitude of the thrust can be given. We consider a plate, located so that, without flapping, the angle of attack equals  $\theta$ . Then, in lowering the plate, the angle of attack is  $\theta + \arctg \frac{w}{V}$ . From this, the force acting on the plate, according to the Zhukovskiy theorem, equals

$$\pi \rho (V^2 + w^2) b \sin \left( \theta + \arctg \frac{w}{V} \right).$$

Since this is force P perpendicular to the resulting velocity it has component  $R_1$  directed forward. This is the thrust (Fig. 22). It is determined by the formula

$$R_1 = \pi \rho (V^2 + w^2) b \sin \left( \theta + \arctg \frac{w}{V} \right) \sin \arctg \frac{w}{V}.$$

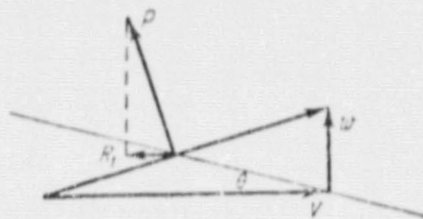


Fig. 22.

In exactly the same way, in raising the wing, we obtain the force directed backward

$$R_2 = -\pi \rho (V^2 + w^2) b \sin \left( \theta - \arctg \frac{w}{V} \right) \sin \arctg \frac{w}{V}.$$

Since forces  $R_1$  and  $R_2$  act during the same time  $T/2$ , their effect in the generation of impulse and the momentum connected with it is determined by the formula

$$R = \frac{R_1 + R_2}{2} = \pi \rho (V^2 + w^2) b \left( \sin \arctg \frac{w}{V} \right)^2 \cos \theta,$$

i.e.,

$$R = \pi \rho b w^2 \cos \theta. \quad (11)$$

By comparing R, calculated by formula (11), with the mean thrust found above

$$X_1 = -\pi \rho b w V \cos \theta \frac{2M-1}{M} \left( \frac{w}{V} - 2 \frac{h}{l} \frac{M-1}{2M-1} \right),$$



where  $M = \frac{h^2}{l} \frac{w}{h}$ , we show that  $X_0 < R$ , under conditions close to the stability condition of a street.

Actually, we formulate the difference

$$V \frac{2M-1}{M} \left( \frac{w}{V} - 2 \frac{h}{l} \frac{M-1}{2M-1} \right) - w = V \frac{M-1}{M} \left( \frac{w}{V} - 2 \frac{h}{l} \right).$$

Since  $\frac{w}{V} > 2 \frac{h}{l}$ , this difference is positive with  $M > 1$ , which, as we have seen, occurs with  $\frac{h}{l} > 0.38$ , i.e., extremely far from satisfaction of the stability condition, when  $\frac{h}{l} = 0.281$ . Thus, formula (11) gives an overstated thrust, which is understandable, since the departing vortices, as it is easy to see, decrease the change in the wash. /551

Calculation of the momentum of the fluid passing through the control surface permits avoidance of those difficulties which arise in accounting for the direct effect of the departing vortices on the velocity around the wing.

## 12. Flight Conditions

We consider the center of gravity of an aircraft with a flapping wing, and we clarify the question, as to the conditions under which this center of gravity returns to the same altitude above the earth at the end of the oscillation period  $T$ .

The weight of the aircraft  $mg$  acts on the center of gravity, where  $m$  is the mass of the aircraft directed downward and, besides, the lift caused by the flapping of the wing directed upward, which changes its magnitude from  $P_1 = mj_1$ , corresponding to lowering the wing, to  $P_2 = mj_2$ , which corresponds to raising the wing. For flight in the case we are considering, it is necessary that  $mj_1 > mg > mj_2$ .

Initially, we consider those flight conditions, when the center of gravity is raised upward from the lowest point. First, this corresponds to lowering the wing and, second, to a certain interval of time, when the wing is rising upward, but the center of gravity, because of inertia, still continues to move upward.

With the  $x$  axis directed vertically upward for the first interval of time, we obtain the equation of motion in the form

$$\frac{d^2x}{dt^2} = j_1 - g, \quad (1)$$

from which, on consideration that  $x_0 = 0$  and velocity  $v_0 = 0$  at the lowest point, for this first interval of time, we obtain

$$\frac{dx}{dt} = (j_1 - g)t,$$

$$x = (j_1 - g) \frac{t^2}{2}.$$

Let the interval of time during which raising of the center of gravity occurs under these conditions be  $T_1$  ( $T_1 < \frac{T}{2}$ ), then, at the end of interval  $T_1$ , the rise of the center of gravity is

$$h_1 = (j_1 - g) \frac{T_1^2}{2} \quad (2)$$

and the velocity is

$$v_1 = (j_1 - g) T_1$$

During that interval of time in which the wing rises, by inertia, the center of gravity also still continues to move upward, the equation of motion of the center of gravity is

$$\frac{d^2x}{dt^2} = -(g - j_2), \quad (3)$$

with initial values  $v_1 = (j_1 - g) T_1$  and  $h_1 = (j_1 - g) \frac{T_1^2}{2}$ . From Eq. (3) and the initial data, we obtain

$$\frac{dx}{dt} = -(g - j_2)t + (j_1 - g) T_1,$$

$$x = -(g - j_2) \frac{t^2}{2} + (j_1 - g) T_1 t + (j_1 - g) \frac{T_1^2}{2}.$$

At the highest point of the rise of the center of gravity, we obtain

$$-(g - j_2)t + (j_1 - g) T_1 = 0,$$

and, by designating the duration of this second condition of rising of the center of gravity by  $\tau_1$ , we obtain

$$\tau_1 = \frac{j_1 - g}{g - j_2} T_1 \quad (4)$$

and, for the corresponding rise  $H_1$  during the entire raising condition,

$$H_1 = -(g - j_2) \left( \frac{j_1 - g}{g - j_2} \right)^2 \frac{T_1^2}{2} + (j_1 - g) \frac{j_1 - g}{g - j_2} T_1^2 + (j_1 - g) \frac{T_1^2}{2},$$

i.e.,

$$H_1 = \frac{j_1 - g}{g - j_2} (j_1 - j_2) \frac{T_1^2}{2}.$$

and, for the total duration of the rise

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$$T_1 + \tau_1 = \frac{h - j_2}{g - j_2} T_1 \quad (5)$$

With the total duration of the rise designated  $\sigma_1$ , we have

$$H_1 = \frac{g(j_1 - j_2)(\sigma_1 - T_1)^2}{2} \quad (6)$$

In a similar manner, we consider the second period, the period of descent of the center of gravity. This period also consists of two parts: first, when Eq. (2) occurs and, second, when Eq. (1) occurs.

From Eq. (3) and initial conditions  $x=H_1$ ,  $v=0$ , for the duration of this second condition  $T_2$ , we have

$$\begin{aligned} \frac{dx}{dt} &= -(g - j_2)t, \\ x &= -(g - j_2) \frac{t^2}{2} + H_1. \end{aligned}$$

Consequently, at time  $T_2$ , we have

$$\begin{aligned} v_1 &= -(g - j_2) T_2, \\ x_1 &= -(g - j_2) \frac{T_2^2}{2} + H_1. \end{aligned}$$

Subsequently, motion occurs by the equation

$$\frac{d^2x}{dt^2} = j_1 - g$$

with initial given  $x_1$  and  $v_1$ . Consequently,

$$\begin{aligned} \frac{dx}{dt} &= (j_1 - g)t - (g - j_2) T_2, \\ x &= (j_1 - g) \frac{t^2}{2} - (g - j_2) T_2 t - (g - j_2) \frac{T_2^2}{2} + H_1. \end{aligned}$$

If the duration of this condition is  $\tau_2$  and, at the end of it, the center of gravity returns to the initial state, we have

$$\begin{aligned} 0 &= (j_1 - g) \tau_2 - (g - j_2) T_2, \\ 0 &= (j_1 - g) \frac{\tau_2^2}{2} - (g - j_2) T_2 \tau_2 - (g - j_2) \frac{T_2^2}{2} + H_1. \end{aligned}$$

from which

$$\left. \begin{aligned} \tau_2 &= \frac{g-j_2}{j_1-g} T_2, \\ H_1 &= \frac{(g-j_2)(j_1-j_2)}{j_1-g} \frac{T_2^2}{2}. \end{aligned} \right\} \quad (7)$$

If the total time of descent of the center of gravity  $\tau_2$  is introduced, we obtain

$$\sigma_2 = T_2 + \tau_2 = \frac{j_1-j_2}{j_1-g} T_2 \quad (8)$$

and

$$H_1 = \frac{(g-j_2)(j_1-g)}{j_1-j_2} \frac{\sigma_2^2}{2}, \quad (9)$$

i.e., the equality is completely analogous to that obtained above. From Eq. (5) and (9), we have

$$\sigma_1 = \sigma_2$$

i.e., the rise time and descent time of the center of gravity are equal.

If the period of oscillation of the wing is  $T$ , according to the preceding,

$$\tau_1 + T_2 = \frac{T}{2} \quad \text{or} \quad \tau_2 + T_1 = \frac{T}{2},$$

from which, by substituting the values of  $\tau_1$  and  $\tau_2$  from (4) and (7), we have

$$\frac{j_1-g}{g-j_2} T_1 + T_2 = \frac{g-j_2}{j_1-g} T_2 + T_1,$$

or

$$\frac{j_1+j_2-2g}{g-j_2} T_1 + \frac{j_1+j_2-2g}{j_1-g} T_2 = 0,$$

or, otherwise,

$$(j_1+j_2-2g) \frac{T_1}{j_1-j_2} = 0$$

and, consequently,

$$\frac{j_1+j_2}{2} = g \quad \text{or} \quad \frac{P_1+P_2}{2} = mg. \quad (10)$$

The result obtained expresses the following conclusion:

For flight at constant altitude, in the case of a flapping wing, the average lift has to equal the weight of the aircraft.



We calculate how the position of the center of gravity of the aircraft changes during lowering of the wing. It is evident that it equals the difference in paths covered during intervals  $T_1$  and  $T_2$ . By designating this path  $h$ , from formula (2), we obtain

$$h = (J_1 - g) \frac{T_1^2}{2} + \left\{ (J_1 - g) \frac{T_2^2}{2} - (g - J_2) T_2 T_1 \right\},$$

or, by substituting  $T_2$  from (7),

$$h = (J_1 - g) \frac{T_1^2}{2} + \left\{ (J_1 - g) \frac{1}{2} \left( \frac{g - J_2}{J_1 - g} \right)^2 T_1^2 - (g - J_2) \frac{g - J_2}{J_1 - g} T_1^2 \right\},$$

i.e.,

$$h = \frac{(J_1 - g)^2 T_1^2 - (g - J_2)^2 T_1^2}{2(J_1 - g)},$$

or, finally, from Eq. (5) and (8),

$$h = 0$$

i.e., at the beginning and end of the wing lowering interval, its center of gravity is at the same altitude.

It is evident that the same is valid for the wing raising interval. Thus, on the average, it is as if the center of gravity is at the same altitude and that the wing shifts downward and upward from this average position.

### 13. Maximum Thrust of Wing

The lift of a wing is determined, as is known, from the formula

$$P = \rho V \Gamma l,$$

where  $l$  is the wingspan or, by substituting the value of  $\Gamma$ ,

$$P = \rho V^2 l \pi b \sin \left( \frac{\alpha}{2} + \theta \right).$$

Let the wing profile permit flow past without detachment of the stream to angle  $\theta_1$ , so that the maximum possible lift of the wing is

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$$P_{\max} = \pi \rho l b V^2 \sin \left( \frac{\alpha}{2} + \theta_1 \right).$$

On the other hand, to obtain the lift of the wing equal to the weight of the aircraft, let it be sufficient to have angle of attack  $\theta_0$ , so that

$$mg = \pi \rho l b V^2 \sin \left( \frac{\alpha}{2} + \theta_0 \right).$$

From the results of the preceding section, if the lift fluctuates from the maximum  $P_1$  to the minimum  $P_2$  during flapping of the wing, for flight at the same altitude, the condition

$$P_1 + P_2 = 2mg;$$

should be satisfied, from which, with the angle of attack which corresponds to the minimum lift  $P_2$  in the case under consideration designated  $\theta_2$ , we obtain the equation

$$\sin\left(\frac{\alpha}{2} + \theta_1\right) + \sin\left(\frac{\alpha}{2} + \theta_2\right) = 2 \sin\left(\frac{\alpha}{2} + \theta_0\right). \quad (1)$$

Since, because of the smallness of the angles, in place of (1),

$$\theta_1 + \theta_2 = 2\theta_0, \quad (2)$$

can be written, from which, from a given  $\theta_1$ , we find  $\theta_2$ .

Let the angles of attack relative to the flight direction in lowering and raising the wing have the values  $\theta_1$  and  $\theta_2$ , respectively, so that

$$\begin{aligned} \theta_1 + \arctg \frac{w}{V} &= \theta_1, \\ \theta_2 - \arctg \frac{w}{V} &= \theta_2 = 2\theta_0 - \theta_1. \end{aligned} \quad (3)$$

From these equations, we have

$$\theta_2 - \theta_1 = 2 \arctg \frac{w}{V} - (\theta_1 - \theta_2) = 2 \left[ \arctg \frac{w}{V} - \frac{\theta_1 - \theta_2}{2} \right].$$

As we have seen, from the conditions of stability of a street, it /557 turns out that  $\frac{w}{V} > 0.562$ , so that  $\arctg \frac{w}{V} > 30^\circ$ . On the other hand, the order of magnitude of  $\theta_1$  and  $\theta_2$  of modern wings at adequate flight speeds are  $15^\circ$  and  $5^\circ$ , respectively. Consequently,  $\theta_2 > \theta_1$  and, in this case,  $\theta_1$  is negative and  $\theta_2$  is positive. Thus, we obtain this result:

In lowering a wing, the angle of attack is less than in raising the wing, in which, in lowering the wing, the angle of attack is negative and, in raising, it is positive.

It is possible that this explains the phenomenon observed in the flight of birds, when it appears that a bird, in lowering the wing, seems to scrape, drive the air backwards.

From the equalities found, it is extremely easy to find the maximum thrust of a wing. For this, we find the maximum circulation  $\gamma$  of

the vortices of the street produced, which are shed from the wing. Since, in lowering the wing and in raising it, the circulations are, respectively,

$$\Gamma_1 = -\pi \sqrt{V^2 + \omega^2} b \sin \theta_1,$$

$$\Gamma_2 = -\pi \sqrt{V^2 - \omega^2} b \sin \theta_2,$$

then,

$$\Gamma = \Gamma_2 - \Gamma_1 = 2\pi \sqrt{V^2 + \omega^2} b \sin \frac{\theta_1 - \theta_2}{2} \cos \frac{\theta_1 + \theta_2}{2}$$

and, from the preceding,

$$\frac{\theta_1 + \theta_2}{2} = \theta_0 \text{ и } \frac{\theta_1 - \theta_2}{2} = \theta_1 - \theta_0.$$

Consequently,

$$\Gamma = 2\pi \sqrt{V^2 + \omega^2} b \sin (\theta_1 - \theta_0) \cos \theta_0.$$

On the other hand

$$X_0 = -\rho V \Gamma \frac{2 \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1}{2 \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l}} \left\{ \frac{\omega}{V} - 2 \frac{h}{l} \frac{\frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1}{2 \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1} \right\},$$

i.e., with a given  $h/l$ , the average thrust is proportional to circulation  $\Gamma$ . Finally, from this, we obtain the maximum thrust in the form /558

$$X_0 = -\rho V 2\pi \sqrt{V^2 + \omega^2} b \sin (\theta_1 - \theta_0) \cos \theta_0 \times \\ \times \frac{2 \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1}{2 \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l}} \left\{ \frac{\omega}{V} - 2 \frac{h}{l} \frac{\frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1}{2 \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1} \right\}.$$

We consider the equation

$$\frac{b}{h} = \frac{\frac{\omega}{V} - 2 \frac{h}{l}}{2 \frac{h}{l} \left[ \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} \right] \cos \tau \left[ \sin \delta + \cos \delta \frac{\omega}{V} \right]};$$

noting that

$$\sin \delta + \cos \delta \frac{\omega}{V} = \frac{1}{V} \sqrt{V^2 + \omega^2} \sin \left( \delta + \operatorname{arctg} \frac{\omega}{V} \right)$$

and

$$\delta = \frac{\theta_1 - \theta_2}{2} = \frac{\theta_1 - \theta_2}{2} - \operatorname{arctg} \frac{\omega}{V},$$

so that

$$\delta + \operatorname{arctg} \frac{\omega}{V} = \frac{\theta_1 - \theta_2}{2} = \theta_1 - \theta_0, \quad \tau = \frac{\theta_1 + \theta_2}{2} = \frac{\theta_1 + \theta_2}{2} = \theta_0,$$

we have

$$\frac{b}{h} = \frac{\frac{w}{V} - 2 \frac{h}{l}}{2 \frac{h}{l} M \cos \theta_0 \sin(\theta_1 - \theta_0) \sqrt{1 + \frac{w^2}{V^2}}}.$$

From this, we determine  $w/V$ . We have the quadratic equation

$$\begin{aligned} \frac{w^2}{V^2} \left[ 1 - \left( 2 \frac{b}{l} M \cos \theta_0 \sin(\theta_1 - \theta_0) \right)^2 \right] - 4 \frac{h}{l} \frac{w}{V} + 4 \frac{h^2}{l^2} - \\ - 4 \frac{b^2}{l^2} M^2 \cos^2 \theta_0 \sin^2(\theta_1 - \theta_0) = 0, \end{aligned}$$

and, since the value of  $2 \frac{h}{l} M \cos \theta_0 \sin(\theta_1 - \theta_0)$  is considerably less than unity, from this, we have approximately

$$\frac{w}{V} = 2 \frac{h}{l} \pm \sqrt{4 \frac{h^2}{l^2} - 4 \frac{h^2}{l^2} + 4 \frac{b^2}{l^2} M^2 \cos^2 \theta_0 \sin^2(\theta_1 - \theta_0)}.$$

Since  $\frac{w}{V} > 2 \frac{h}{l}$ , finally,

$$\frac{w}{V} = 2 \frac{h}{l} \left\{ 1 + \frac{b}{h} \left( \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} \right) \cos \theta_0 \sin(\theta_1 - \theta_0) \right\}. \quad (4)$$

By substituting this value in formula (17'''), Section 10, we obtain the final expression of the maximum thrust of the wing

$$\begin{aligned} X_0 = -\pi \rho V^2 l \left\{ \frac{b}{h} \left( \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} \right) \cos \theta_0 \sin(\theta_1 - \theta_0) \right\} \times \\ \times \left\{ \frac{b}{h} \cos \theta_0 \sin(\theta_1 - \theta_0) \left( 2 \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1 \right) + 2 \frac{h}{l} \right\} \frac{1}{2 \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l}}, \end{aligned}$$

i.e.,

$$\begin{aligned} X_0 = -\frac{\pi}{2} \rho V^2 b \cos \theta_0 \sin(\theta_1 - \theta_0) \times \\ \times \left[ \frac{b}{h} \cos \theta_0 \frac{2 \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1}{\frac{h}{l}} \sin(\theta_1 - \theta_0) + 2 \right], \end{aligned}$$

which, further, can be written approximately in the form

$$X_0 = -\frac{\pi}{2} \rho V^2 b (\theta_1 - \theta_0) \left[ 2 + \frac{b}{h} \frac{l}{h} \left( 2 \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l} - 1 \right) (\theta_1 - \theta_0) \right]. \quad (5)$$

If the values which correspond to the stability condition are inserted here, we obtain

$$X_0 = -\frac{\pi}{2} \rho V^2 b (\theta_1 - \theta_0) \left[ 2 + 0.91 \frac{b}{h} (\theta_1 - \theta_0) \right], \quad (6)$$

$$X_0 = -\rho V^2 b (\theta_1 - \theta_0) \left[ 3.14 + 1.43 \frac{b}{h} (\theta_1 - \theta_0) \right], \quad (7)$$



or approximately, in view of the smallness of  $\theta_1 - \theta_0$ ,

$$X_0 \approx -\pi \rho V^2 b (\theta_1 - \theta_0). \quad (8)$$

It is of interest to compare the resulting formula with the formula for the maximum lift of a stationary wing

$$Y_{\max} = \pi \rho V^2 b \left( \frac{\alpha}{2} + \theta_1 \right).$$

From Eq. (8), we have

$$X_{\max} = -\pi \rho V^2 b \left[ \left( \theta_1 + \frac{\alpha}{2} \right) - \left( \theta_2 + \frac{\alpha}{2} \right) \right] 2 \frac{h}{l} = -[Y_{\max} - Y_0] 0.562. \quad (9)$$

By analogy with theories of ships, expression  $Y_{\max} - Y_0$  can be called the lift reserve of the wing and, consequently, Eq. (9) gives the following result:

The maximum thrust of a flapping wing equals its lift reserve multiplied by  $2 \frac{h}{l}$ .

From this, in particular, it follows that flight at constant altitude is possible, only if, at a given flight speed, the wing has a lift reserve. Since, in constant speed flight, the thrust of a flapping wing is used to overcome drag, from this, we also find that flight at constant altitude is possible, only in the event the drag of the aircraft is less than the lift reserve of the wing.

Let the coefficient of drag of the aircraft be  $c_Q$ . Then, the drag per unit of wingspan equals  $c_Q \rho V^2 b$  and, consequently, by comparison with (9), for the possibility of flight at constant altitude, we have the inequality

$$c_Q < 0.562\pi (\theta_1 - \theta_0) = 1.76 (\theta_1 - \theta_0).$$

#### 14. Optimum Flight Conditions; Examples

If flight conditions are considered, in which the Karman stability condition is satisfied, the preceding results permit it to be shown that the number of flaps at given  $V$  and  $w$  is connected with the chord dimensions and with the flight speed  $V$ , by an extremely simple relationship.

Actually, as formulas (12) and (13), Section 4, show, in flight

in the conditions under consideration, the following condition should be satisfied

$$\frac{w}{V} > 0,562. \quad (1)$$

On the other hand, the value of the  $w/V$  ratio permits, with given  $\theta_1$  and  $\theta_2$ , from which the values of  $\alpha = \frac{\theta_1 + \theta_2}{2}$  and  $\beta = \frac{\theta_1 - \theta_2}{2}$ , now are found, determination of the value of  $b/h$  by the formula

$$\frac{b}{h} = \frac{\frac{w}{V} - 0,562}{0,35 \cos \alpha \left[ \sin \beta + \cos \beta \frac{w}{V} \right]}, \quad (2)$$

from which we find  $l$ , by using the ratio

$$\frac{h}{l} = 0,281. \quad (3)$$

From ratios (2) and (3), we find

$$l = \frac{1,246 \cos \alpha \left[ \sin \beta + \cos \beta \frac{w}{V} \right]}{\frac{w}{V} - 0,562} b. \quad (4)$$

Since

$$u_0 = \frac{b}{h} \cos \alpha \left[ \sin \beta + \cos \beta \frac{w}{V} \right] V \frac{h\pi}{l} \operatorname{th} \frac{h\pi}{l},$$

by substituting the value of  $b/h$  from Eq. (2), and noting that, at  $\frac{h}{l} = 0,281$ , we have  $\frac{h\pi}{l} \operatorname{th} \frac{\pi}{l} = 0,616$ , further, we obtain

$$u_0 = 1,8 \left( \frac{w}{V} - 0,562 \right) V. \quad (5)$$

If the number of flaps per second is designated  $N$ ,

$$N = \frac{1}{T} = \frac{V + u_0}{l},$$

from which, by substituting the values of  $l$  and  $u_0$  from Eq. (4) and (5), we obtain

$$N = \frac{\left[ 1 + 1,8 \left( \frac{w}{V} - 0,562 \right) \right] \left[ \frac{w}{V} - 0,562 \right]}{1,246 \cos \alpha \left( \sin \beta + \cos \beta \frac{w}{V} \right)} \frac{V}{b}. \quad (6)$$

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Quantities  $w/V$ ,  $\alpha$  and  $\delta$  characterize the flight conditions during flapping. Therefore, the quantity /562

$$K = \frac{\left[1 + 1.8 \left(\frac{w}{V} - 0.562\right)\right] \left[\frac{w}{V} - 0.562\right]}{1.216 \cos \alpha \left[\sin \frac{\delta}{2} + \cos \frac{\delta}{2} \frac{w}{V}\right]} \quad (7)$$

can be called the flight conditions coefficient. By substitution of  $K$  in Eq. (6), we obtain

$$N = K \frac{V}{b}. \quad (8)$$

Formula (8) shows that, in the same flight conditions (with equal  $K$ ), the number of oscillations is directly proportional to the flight speed and inversely proportional to the size of the chord.

By comparing formulas (7) and (2), for coefficient  $K$ , we obtain the expression

$$K = 0.281 \left[1 + 1.8 \left(\frac{w}{V} - 0.562\right)\right] \frac{b}{h}, \quad (9)$$

and here, if  $(\frac{w}{V} - 0.562)$  is small, as usually is the case, approximately, we have

$$K = 0.281 \frac{b}{h}. \quad (10)$$

It is evident that formula (10) is obtained from the expression

$$N = \frac{V \left(1 + \frac{u_0}{V}\right)}{l},$$

if the term  $u_0/V$  in it is disregarded. Thus, further, we obtain an expression for  $N$  in the form

$$N = 0.281 \frac{b}{h}, \quad \frac{V}{b} = 0.281 \frac{V}{h}. \quad (11)$$

We consider the following example.

The critical angle of a flapping profile  $\theta_{\max} = 16^\circ$ . The flight condition is determined from the following data /563

$$\arcsin \frac{w}{V} = \frac{\pi}{6}, \quad \frac{w}{V} = 0.577, \quad \alpha = 0;$$

the angles of attack in lowering the wing ( $\theta_1$ ) and in raising the wing ( $\theta_2$ ) are determined from the conditions

$$\begin{aligned}\frac{\alpha}{2} + \theta_1 + \arctg \frac{w}{V} &= \theta_{\max}, \\ \frac{\alpha}{2} + \theta_2 - \arctg \frac{w}{V} &= 0,\end{aligned}$$

from which  $\theta_1 = -14^\circ$ ,  $\theta_2 = 30^\circ$  and, consequently,

$$\varepsilon = 8^\circ, \quad \delta = -22^\circ;$$

1. we find  $b/h$ :

$$\frac{b}{h} = \frac{0,577 - 0,562}{0,35 \cdot 0,99 [-0,3746 + 0,9272 \cdot 0,577]} = 0,27;$$

2. we find the mean circulation  $\Gamma$ :

$$\Gamma = -\pi b V \sin \varepsilon \left[ \cos \delta - \sin \delta \frac{w}{V} \right],$$

i.e.,

$$\Gamma = \pi b V 0,1392 [0,9772 + 0,3746 \cdot 0,577] = -\pi b V 0,165;$$

from which we obtain the mean lift:

$$Y_0 = -\rho V \Gamma = 0,5 \rho V^2 b;$$

3. for determination of the number of flaps  $N$ , we use formula (8), from which

$$K = 0,0777,$$

and, therefore

$$N = 0,0777 \frac{V}{b};$$

4. finally, for determination of the thrust, we use formula (15'), Section 9,

$$X_0 = -0,94 \rho b V^2 \cos \varepsilon \left( \sin \delta + \cos \delta \frac{w}{V} \right) \left( 1 + 1,2 \frac{w}{V} \right);$$

by substituting the values found, we obtain

$$X_0 = -0,258 \rho V^2 b,$$

so that the thrust is almost half the lift.

As particular examples, we consider the following.

1. The load of a flight vehicle per unit area  $p = 20 \text{ kg/m}^2$ ,  $b = 3 \text{ m}$ .



We determine the velocity and the number of flaps for flight at constant altitude;<sup>3</sup> from  $Y_0 l = 0.5 \rho V^2 b l$  ( $l$  is the span), we have

$$\rho = \frac{Y_0}{b} = 0.5 \rho V^2 \text{ and } V = \sqrt{\frac{\rho}{0.5 \rho}} = \sqrt{320} = 18 \text{ m/sec,}$$

$$N = 0.0777 \frac{V}{b} = 0.47.$$

2. The same question, with  $p = 5 \text{ kg/m}^2$ ,  $b = 0.1 \text{ m}$ :

$$V = \sqrt{\frac{5}{0.5 \rho}} = 7 \text{ m/sec,}$$

$$N = 0.0777 \frac{V}{b} = 5.4.$$

The data approximately correspond to the flight of a pigeon. Observations gave<sup>4</sup> from 4 to 10 for  $N$  and from 13 to 30 m/sec for  $V$ .

3. The same question<sup>5</sup> with  $p = 0.66 \text{ kg/m}^2$ ,  $b = 0.0002 \text{ m}$ :

$$V = 3.2 \text{ m/sec, } N = 123$$

The data approximately correspond to the flight of a fly. Observations give  $N$  from 180 to 350.

As a second example, we consider the performance of a flapping wing with the following data. The critical angle of attack of the profile is  $15^\circ$ ,  $\alpha = 0$ . The flight conditions are determined from the following data:  $\arctg \frac{w}{V} = 0.6$  ( $31^\circ$ ) and the angles of attack of the wing are determined from the equations

$$\theta_1 + 31^\circ = 15^\circ, \quad \theta_1 = -16^\circ, \quad \sigma = 10^\circ, \quad \delta = -26^\circ,$$

$$\theta_2 - 31^\circ = 5^\circ, \quad \theta_2 = 36^\circ,$$

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<sup>3</sup> The data are similar to those for the ornithopter of A. Soltau (from M.K. Tikhomirov *Polet ptits i mashiny s mashushchimi kryl'yami* [Flight of Birds and Machines with Flapping Wings], ONTI, 1937, p. 104). The vehicle did not fly (see further).

<sup>4</sup> Data from M.K. Tikhomirov, p. 19, 28, 80.

<sup>5</sup> Data from M.K. Tikhomirov, p. 67, 68, 70. In evaluating the figures obtained, it must be remembered that the mechanism of flapping differs in this work and in the data of M.K. Tikhomirov.

From these data:

1.  $\frac{b}{h}=1.1$ ,
2.  $\Gamma=-0.634$  bV,
3.  $Y_0=0.634\rho V^2 b$ ,
4.  $K=0.33$ ,
5.  $X_0=0.16\rho V^2 b$ .

As applied to the particular cases, with the data used above and our conclusions, we obtain:

1. with  $p=20$  kg/m<sup>2</sup>,  $b=3$  m, velocity  $V$  and the number of flaps at constant altitude  $N$  are:

$$V=31 \text{ m/sec, } N=3.3;$$

2. with  $p=5$  kg/m<sup>2</sup>,  $b=0.1$  m,

$$V=2.8 \text{ m/sec, } N=533;$$

3. with  $p=0.66$  kg/m<sup>2</sup>,  $b=0.002$  m,

$$V=16 \text{ m/sec, } N=2640.$$

Comparison of the results obtained with the observations presented above shows that the flight conditions of the Soltau ornithopter are more like the second set of data since, in the first case, we have the absurdly large

$$h=\frac{b}{0.87}=11 \text{ m.}$$

We note that, with  $N=1.5$ , the vehicle did not fly (in the second case,  $h=\frac{b}{1.1}=2.7$  m). The flight of a pigeon and the flight of a fly are closer to the first scheme since, in the first case, for the pigeon, we have  $h=\frac{b}{0.27}=0.4$  m (in the second scheme,  $h=0.1$  m). In just the same way, in the case of the flight of a fly, it is more like the first scheme, /566 where  $h=\frac{0.002}{0.27}=\frac{2}{0.27}$  mm=7.4 mm (by the second scheme,  $h=2$  mm). However, it must be remembered that all data on the flight of birds and insects are extremely uncertain. The resulting conclusions, concerning data of extremely varied dimensions (ornithopter, bird, insect), which are sufficiently consistent with reality, evidently confirm the applicability of the resulting conclusions to actual observations of cases of flight with flapping wings.

In conclusion, we note that, in both schemes, the factor  $\cos \alpha - \sin \alpha \frac{W}{V}$  is greater than one and, because of this, the lift coefficient of a flapping wing is higher than that of a stationary wing at an average angle of attack. Thus, in the first scheme, the lift coefficient corresponds to a  $9^\circ$  angle of attack, while  $\alpha = 8^\circ$  and, in the second scheme, the lift coefficient corresponds to a  $11^\circ$  angle of attack, while  $\alpha = 10^\circ$ .

#### 15. Some Further Conclusions<sup>6</sup>

The theory of flapping wings presented in this work has been developed further. Study of the question of the formation and mechanical properties of "bias" streets is undoubted interest. We have shown that the formation of streets the direction of which diverges from the direction of forward movement of the flapping wing is theoretically possible. Such "bias" streets can form in the event the rate of lowering and raising the wing is nonuniform. This case was studied theoretically and experimentally in detail in the work of Ya.Ye. Polonskiy [10]. The experimental results he obtained are reduced to the following.

"With nonuniform flapping of a wing (the lowering and raising rates are different), the vortices shed form a street, the axis of which is inclined to the direction of forward motion. The street diverges in the direction of the higher flapping speed, and it does not have a checkerboard arrangement of the vortices. In an upward deviation of the street, the vortices of the upper row are shifted to the right of the checkerboard arrangement. In downward deviation of the street from the direction of forward motion, the vortices of the upper row are shifted to the left of the checkerboard arrangement. /567

"The experiment gave one unexpected result. This is the noncheckerboard arrangement of the vortices of the "bias" street, in which, as it turned out, it is of basic importance in examination of such streets."

The latter remark indicates the following. For the formation of a "bias" street, besides velocity in the direction of forward motion of the wing, the street must also have a velocity directed perpendicular to the direction of forward motion of the wing. This "drift" of the street is possible, only with a street structure which is intermediate between the checkerboard and parallel arrangement of the vortices. The question arises here as to the stability of such streets since, in all theories reported, the Karman stability condition plays a significant part. V.A. Ivanova showed that, in the case of such drifting streets, there is the following stability condition, in the sense of Karman:<sup>7</sup>

<sup>6</sup>

See conclusion in article of V.V. Golubev, "Thrust of a Flapping Wing," Izv. AN SSSR OTN 5, (1946) [see this publication, pp. 1].

<sup>7</sup>V.A. Ivanova, "Stability of 'bias' vortex streets," diploma work defended at Mechanical-Mathematical Department, Moscow State University, 1949.

$$\operatorname{ch}^2 \frac{\pi b}{l} + \cos^2 \frac{\pi b}{l} = 2, \quad (1)$$

where  $b$  is the displacement of the vortices of one side of the street relative to the other. In the case of the checkerboard arrangement,  $b=l/2$ , and condition (1) changes to the Karman condition

$$\operatorname{ch}^2 \frac{\pi b}{l} = 2;$$

in the event  $b=0$ , i.e., with parallel arrangement of the vortices, from (1), we have

$$\operatorname{ch}^2 \frac{\pi b}{l} = 1,$$

from which  $b=0$ , i.e., both sides of the street should fuse and, consequently, because of the opposite directions of rotation of the vortices in the two rows of the street, a stable street disappears. This is in complete agreement with the known fact that streets with a parallel vortex arrangement are unstable in the sense of Karman. /568

In his work, Ya.Ye. Polonskiy studied the kinematics and dynamics of "bias" vortex streets in detail. The formulas he obtained are a correlation of the formulas derived above, to which they are converted, in the event the rates of lowering the raising the wing are equal.

Another case which received detailed examination is the case of flight "in place," which can be observed in the flight of small birds and insects. In the work of V.V. Golubev [9], it was shown that this case of flight can be explained, based on formulas (4), (6) and (15), Section 10. Actually, from the equations which determine lift  $Y_0$  and thrust  $X_0$

$$\begin{aligned} Y_0 &= -\rho \pi b V^2 \sin \alpha \left[ \cos \delta - \sin \delta \frac{w}{V} \right], \\ X_0 &= -0.94 \rho b V^2 \cos \alpha \left( \sin \delta + \cos \delta \frac{w}{V} \right) \left[ 1 + 1.2 \frac{w}{V} \right] = \\ &= -0.94 \rho b \cos \alpha (V \sin \delta + w \cos \delta) [V + 1.2w]. \end{aligned}$$

we see that, in the absence of forward velocity, i.e., with  $V=0$ , lift  $Y_0$  reverts to zero, but thrust  $X_0$  differs from zero, since, with  $V=0$ ,

$$X_0 = -1.13 \cos \alpha \cos \delta w^2.$$

Thus, by changing the lift to thrust, for which the body of the bird must be turned from the horizontal position to the vertical, which is observed in the flight of birds "in place," the thrust of the entire bird can be cancelled. In the work mentioned above, a detailed study of this phenomenon is given.

Evidently, the vortex scheme examined here permits determination of the force which acts, not on a flapping wing, but on a rotating



wing. The formation of vortex streets in this case was noted in the work of D.S. Vilker and L.P. Smirnov [14] in 1937.

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